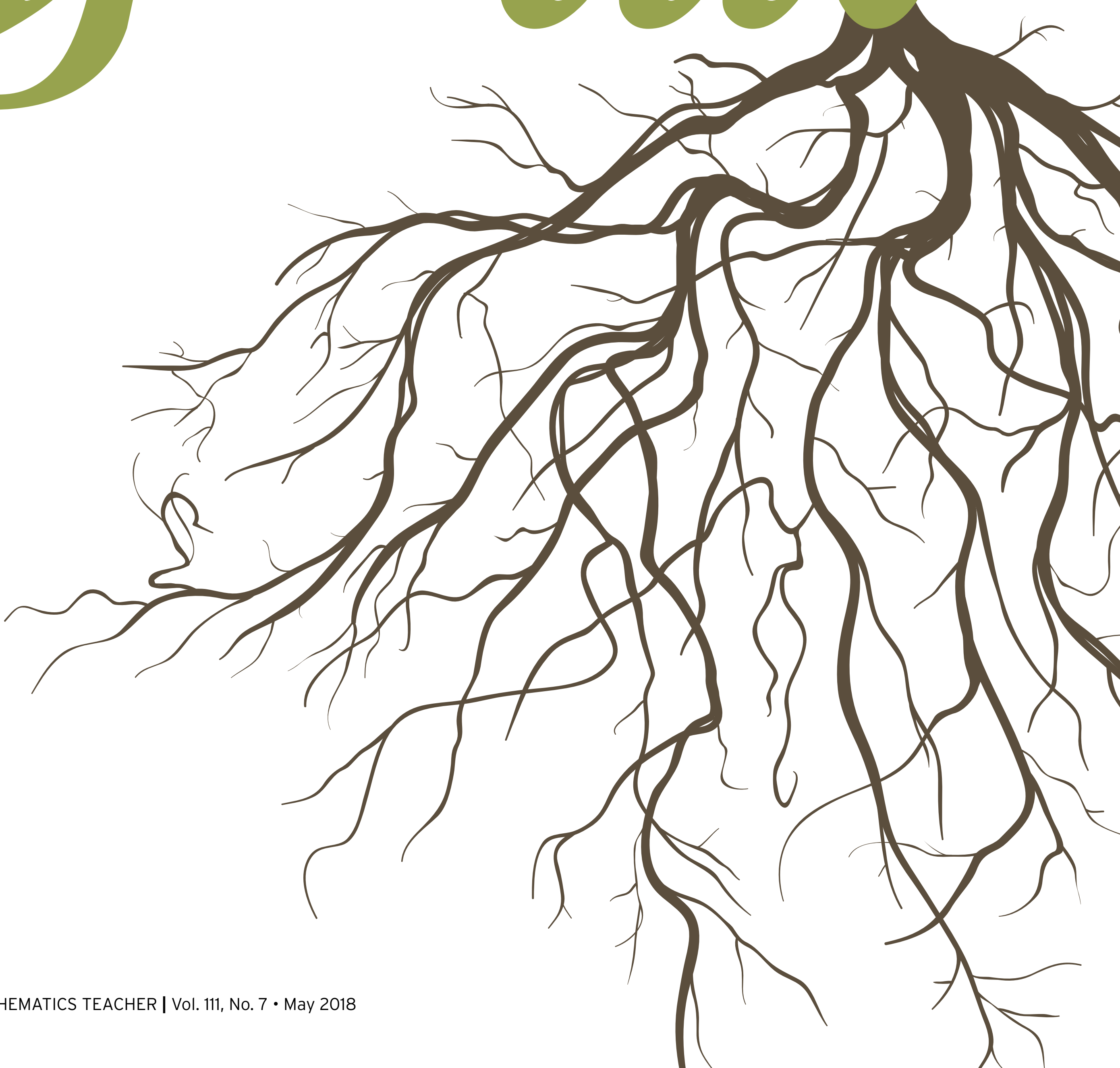
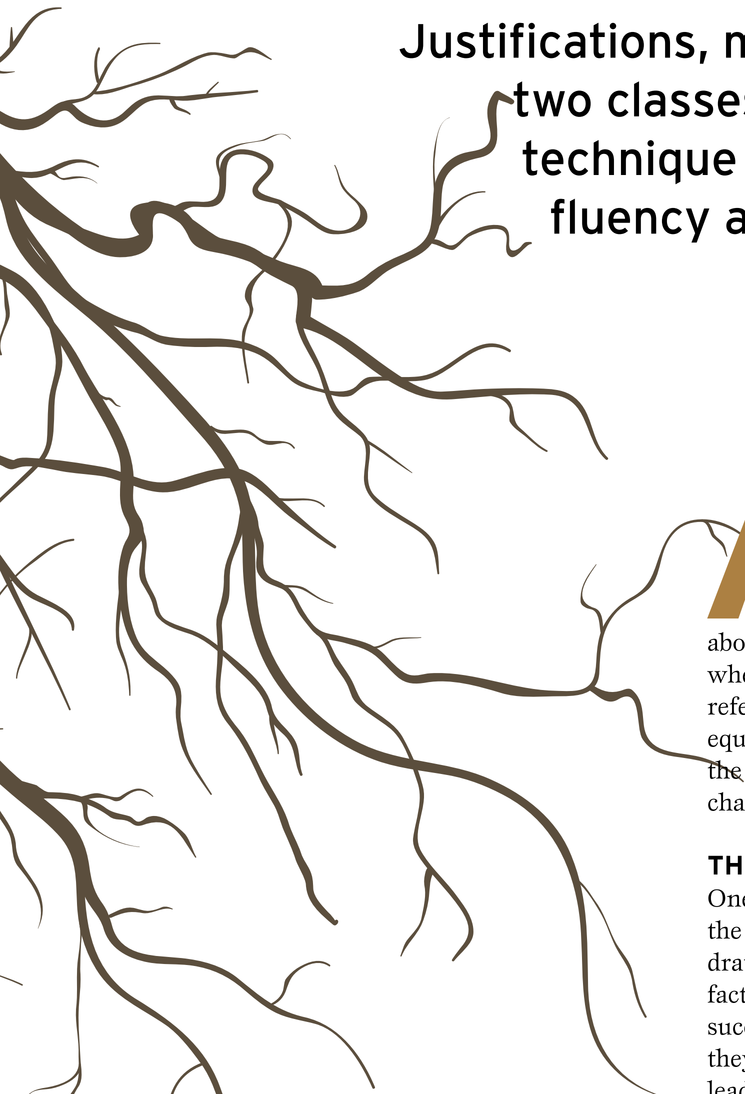
# Factoring for Roots: A Technique to Factor $y = ax^2$



# $+ bx + c$

**Justifications, methods, and results compare two classes of students who used a new technique that ties together procedural fluency and conceptual understanding in a manner unlike other current strategies.**

Adam Clinch



As teachers, we constantly tweak our lessons, making them more appropriate for our students. I considered a new technique for factoring trinomials when I heard questions in class regarding why we would multiply $a$ and $c$ as well as statements about how unconnected doing so was from the prior strategy when $a = 1$. However, it really hit me when my standards-referenced quiz scores for factoring and solving quadratic equations with a leading coefficient not equal to 1 came in and the average score was 69 percent correct. Something needed to change.

## THE ISSUE

One of the foundational topics in first-year algebra concerns the concept of factoring. Students first consider factoring quadratic expressions of the form $x^2 + bx + c$ by searching for two factors of $c$ whose sum is $b$. Generally, students can become successful with these types of problems and explain the steps they are taking. However, the same cannot be said when a leading coefficient other than 1 is introduced into the problem. More explicitly, factoring $ax^2 + bx + c$ with $a \neq 1$ (and going forward $ax^2 + bx + c$ will mean $a \neq 1$) proves to be much more difficult for students. This is perhaps most likely because current textbooks present this topic in a completely different fashion from the way they approach the situation when $a = 1$.

We discuss an alternative strategy for factoring quadratics of the form $ax^2 + bx + c$, known as *factoring for roots*. This strategy enables students to extend the knowledge they used when the leading coefficient was 1 and explain the steps they are taking in solving. Many other current strategies fail in both these endeavors.

## THE EASY STAGE

Traditionally, students begin the study of factoring of quadratics by multiplying binomials such as $(x + 2)(x + 3)$ and producing an answer of $x^2 + 3x + 2x + 6 = x^2 + 5x + 6$. The final term of 6 was found through the product of the original 2 and 3, whereas the middle term with a coefficient of 5 was produced by the sum of 2 and 3. Students see this connection, and it makes the reverse process of producing the factored form of $x^2 + 7x + 12$ straightforward as they search for two numbers whose product is 12 and whose sum is 7. Upon arriving at 3 and 4, they can transform the standard form to factored form and write $(x + 3)(x + 4)$. Although many teachers are able to rely purely on this algebraic demonstration, some use manipulatives, such as Algebra Tiles™ or a modified version of algebra tiles by constructing an area model (see **fig. 1**).

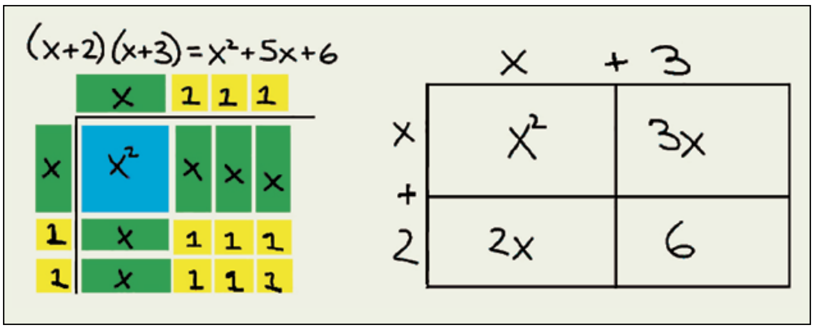



**Fig. 1** Students can use algebra tiles and an area model representation when multiplying binomials, as in this example for $x^2 + 5x + 6$.

However, this concept of factoring quadratics when the leading coefficient is 1 has never been the biggest problem with factoring in the algebra curriculum. Quadratics with a leading coefficient of 1 challenge some students' factoring abilities, but expressions with a leading coefficient not equal to one tend to frustrate even the most proficient.

## PROBLEM: WHEN $a \neq 1$

Factoring expressions of the form $ax^2 + bx + c$ is noticeably more difficult than factoring $x^2 + bx + c$ because there are multiple strategies to attack this new challenge. The lack of consensus in the textbooks' approaches to the setting only compounds this problem. In *Algebra 1: A Common Core Curriculum*, Larson and Boswell suggest making a table to guess and check all the possible factored forms based on the factors of $a$ and $c$ until one also produces the correct value for $b$ (see **table 1**).

**Table 1 An Example of All Possible Factored Forms based on Factors of $a$ and $c$ Until One Also Produces the Correct Value for $b$ for $3x^2 + 7x - 26$**

| Factors of 3 | Factors of –26 | Possible Factorization | Middle Term | |
|---|---|---|---|---|
| 1, 3 | 1, –26 | $(x + 1)(3x - 26)$ | $-26x + 3x = -23x$ | No |
| 1, 3 | 26, –1 | $(x + 26)(3x - 1)$ | $-x + 78x = 77x$ | No |
| 1, 3 | –1, 26 | $(x - 1)(3x + 26)$ | $26x - 3x = 23x$ | No |
| 1, 3 | –26, 1 | $(x - 26)(3x + 1)$ | $x - 78x = -77x$ | No |
| 1, 3 | 2, –13 | $(x + 2)(3x - 13)$ | $-13x + 6x = -7x$ | No |
| 1, 3 | 13, –2 | $(x + 13)(3x - 2)$ | $-2x + 39x = 37x$ | No |
| 1, 3 | –2, 13 | $(x - 2)(3x + 13)$ | $13x - 6x = 7x$ | Yes |
| 1, 3 | –13, 2 | $(x - 13)(3x + 2)$ | $2x - 39x = -37x$ | No |

Adapted from Larson and Boswell 2015, p. 394

I can already sense the audible groans of frustration that a teenage student would generate upon being presented with this strategy. It is cumbersome and tedious for students and unfair to teachers asked to present it. Can you imagine explaining this strategy to an absent student for a problem such as $12x^2 - 23x - 24$? Such a table would consist of 48 rows and $4 \cdot 48 = 192$ cells!

I also know that some teachers discuss the concept of considering all the possibilities for such situations as $2x^2 - 3x - 20$ without committing to paper and pencil. However, I cannot show all students how this mental game plays out because it is done strictly in one's head. I can give them strategies and show example after example, but not all students have obtained the number sense that is required for this mental exercise at this stage of their development. Textbooks recognize this, which is why they present the table method.

McGraw Hill's *Algebra 1* presents two strategies: (1) the same guess-and-check strategy as above and (2) a version of factoring by grouping where one splits the middle term by finding two numbers whose sum is $b$ and whose product is $a \cdot c$. Let's see how this strategy works with $4x^2 - 23x + 15$. We need to find two numbers whose sum is –23 (since we will be splitting the middle term) and whose product is $4 \cdot 15 = 60$. The text explains neither why this strategy works nor where the idea comes from. The two numbers are –20 and –3, and the book suggests a version of the worked-out strategy shown in **figure 2**.

| | |
|---|---|
| Factoring $ax^2 + bx + c$ | |
| Words | To factor quadratic trinomials of the form $ax^2 + bx + c$, one must obtain two integers, $m$ and $n$, with a sum of $b$ and a product of $a \cdot c$. Finally, rewrite $ax^2 + bx + c$ as $ax^2 + mx + nx + c$ and factor by grouping. |
| Example | $4x^2 - 23x + 15 = 4x^2 - 20x - 3x + 15$ (Note: $m = -20$ and $n = -3$)<br>$= 4x(x - 5) - 3(x - 5)$<br>$= (4x - 3)(x - 5)$ |

**Fig. 2** Factoring a quadratic expression by grouping can be perplexing if no justification for the first step is discussed (adapted from Carter et al. 2010, p. 510).

Other versions of this method exist (one well-known example is called Bottoms Up or Slip-Slide Factoring), and I highly recommend Jeffrey Steckroth's (2015) wonderfully informative article on Slip-Slide Factoring from a transformational point of view. However, these techniques all rely on the step where one needs two numbers whose product is $a \cdot c$. The problem is that the justification for this method is both extremely involved for a first-year algebra student to comprehend and some teachers to defend, and it bears little to no resemblance to the strategy when $a$ is 1. Alas, in some texts the justification is completely absent. Mathematics educator John A. Van de Walle and his colleagues (2014, p. 1) state:

> One hallmark of mathematical understanding is a student's ability to justify why a given mathematical claim or answer is true or why a mathematical rule makes sense.

More often than not when procedural fluency trumps conceptual understanding, a student's mathematical comprehension is incomplete.

## PROCEDURAL FLUENCY AND CONCEPTUAL UNDERSTANDING REQUIRED

The NCTM position statement concerning procedural fluency defines it as the—

> ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another.

Unfortunately, once students master a procedural tool without yet acquiring the appropriate conceptual understanding to accompany it, they have little motivation to seek out justification because the tool works (Hiebert 1999). To combat this, the authors of *Adding It Up* (NRC 2001) state

**Factoring expressions with a leading coefficient not equal to one tend to frustrate even the most proficient.**

that conceptual understanding, the "how" and "why" the method works, should be present in all phases of the procedural teaching.

Procedural fluency and conceptual understanding form a symbiotic relationship that promotes mathematical proficiency. The procedure makes the conceptual understanding more efficient and practical, whereas the conceptual understanding justifies the initial usage and possible applications of the procedure outside the original realm (Brownell 1935). In essence, conceptual understanding allows students to connect the new procedure to prior knowledge. NCTM's Connections Standard (NCTM 2000, p. 64) and *Principles to Actions: Ensuring Mathematical Success* (2014) go even further in arguing that for learning to have meaning and stand a chance of being remembered, it must be built on prior knowledge.

Unfortunately, current textbook techniques present an unjustified tool that fails to build on students' understanding of factoring in the $x^2 + bx + c$ setting. That is, it lacks a foundation of true prior knowledge built on conceptual understanding. The following strategy alleviates this situation.

## A SOLUTION: FACTORING FOR ROOTS

Consider the expression $2x^2 - 5x - 12$. We can begin by asking students how this is different from when we factored expressions of the form $x^2 - x - 6$. Students would note that we have a leading coefficient not equal to 1. So, how could we make it look like the previous scenario that we know how to tackle?

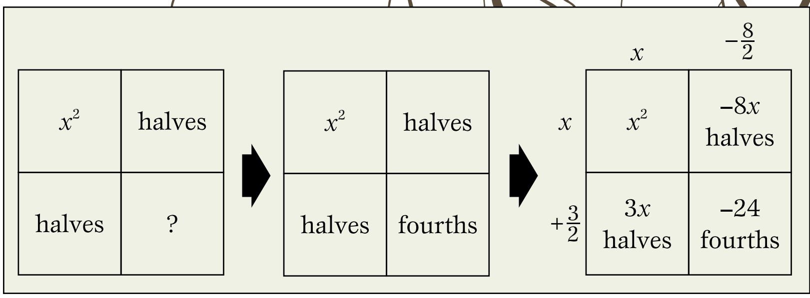



**Fig. 3** Using an area model to factor a quadratic with fractions helps students keep ideas organized.

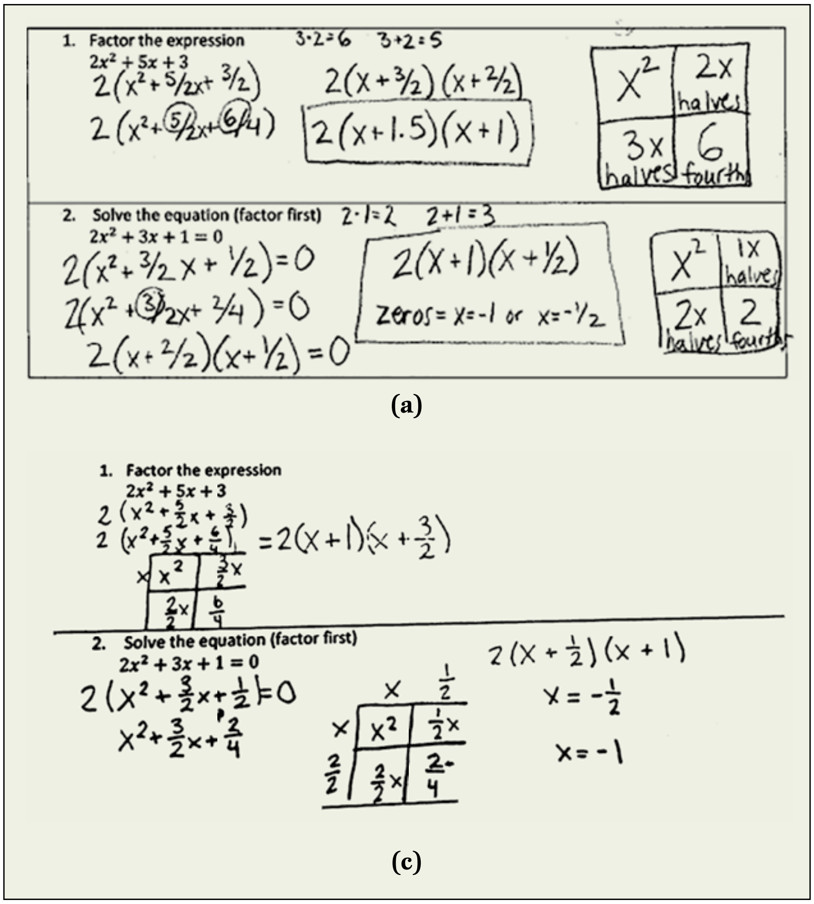



**Fig. 4** Examples of student work as they use the Factoring for Roots method show that they understand the process both procedurally and conceptually.

We could factor out a 2, and now we are at $2(x^2 - 5x/2 - 12/2)$.

We know how to address this because the leading coefficient inside the parentheses is now 1. We need two numbers whose product is $-12/2$ and whose sum is $-5/2$. If they have a sum of $-5/2$, we know that in factored form, it will look like $2(x + ?/2)(x - ?/2)$.

However, when we do so, it becomes apparent that these halves will produce fourths when we reconstruct $c$, so we rewrite $c$ as $-24/4$ and get $2(x^2 - 5x/2 - 24/4)$.

Since we have addressed the denominators, we need consider only the numerators. Therefore, we simply need two numbers whose product is $-24$ and whose sum is $-5$. These would be 3 and $-8$, so our factored form must be the following:

$$2(x + 3/2)(x - 8/2)$$
$$(2x + 3)(x - 4)$$

This process makes clear why we would multiply $a$ and $c$, which is to acquire the appropriate denominator. As with other methods, students will still be required to find two numbers whose sum is $b$ and whose product is $a \cdot c$, for that is at the heart of the interconnection between multiplying binomials and factoring trinomials. What necessitates a search for the product $a \cdot c$ is now evident. Because initially this may be a handful for a first-year algebra student, it might be wise to use the area model strategy (see **fig. 3**). Particularly when working with equations and not just expressions, I would

GLASSCAGE/SHUTTERSTOCK

**Their mathematics was built on a solid foundation of prior knowedge and conceptual understanding.**

not require my students to simplify from $y = 2(x + 3/2)(x - 4)$ to $y = (2x + 3)(x - 4)$.

We certainly discuss this simplification because students will never see anyone leave it as the former, but we also discuss why we care about factoring in the first place, which is to find $x$-intercepts. At this point in the year, my students have already solved such equations as $0 = x^2 - 6x + 8$ by factoring, and they have graphed $y = x^2 - 6x + 8$ by factoring first to find the roots and then using the symmetry of the parabola to find its vertex. They know we factor an equation to find its roots, so the form $y = 2(x + 3/2)(x - 4)$ makes it easier to find them.

To further show the advantages of this method, we consider the problem posed earlier, $y = 12x^2 - 23x - 24$, which would require 48 rows and 192 cells through the guess-and-check method of some current textbooks. First, factor out the 12, and then adjust the final term:

$$y = 12(x^2 - 23x/12 - 24/12)$$
$$y = 12(x^2 - 23x/12 - 288/144)$$

Now, with the denominators accounted for, produce two numbers whose product is –288 and whose sum is –23. One might begin with combinations –144 and 2, –72 and 4, –48 and 6, and –36 and 8 but find that all are too far apart (actually, realizing they need to sum to –23 would eliminate some of these earlier choices for proficient students). The last option of –36 and 8 sums to –28, so it is close. Next, try –32 and 9 and find success:

$$y = 12(x - 32/12)(x + 9/12)$$
$$y = 12(x - 8/3)(x + 3/4)$$

The roots of the quadratic equation are thus 8/3 and –3/4. If desired, one could break 12 into 3 • 4 and distribute the 3 through the first parentheses and the 4 through the second to arrive at $y = (3x - 8)(4x + 3)$.

## DID I SEE IMPROVEMENT?

Before implementing the factoring for roots strategy, my students articulated that they struggled with the fact that two seemingly identical scenarios, factoring when $a$ is or is not 1, would have such different approaches. This case-by-case analysis left them frustrated, and their average quiz score of 69 percent (sample size of 38 and standard deviation of 28.5) on this skill reflected their exasperation.

The next spring, I introduced the technique of factoring for roots. Using the same questions as the previous year with no other additional supports, the quiz scores had an average of 83 percent (with a sample size of 41 and a standard deviation of 21.2). I knew I was on to something. I could see students' work support the steps they were taking (see **fig. 4**). There were no leaps of faith; their mathematics built on a solid foundation of prior knowledge and conceptual understanding.

**Table 2 Factoring a Quadratic Expression by Using Fractions to Find the Roots.**

| Step | Description | Algebraic Steps |
|---|---|---|
| 1 | Consider a quadratic expression $ax^2 + bx + c$. | $4x^2 - 8x - 5$ |
| 2 | Factor out the leading coefficient, $a$, and do not simplify fractions for the middle and final terms. | $4(x^2 - 8/4x - 5/4)$ |
| 3 | For the expression in parentheses, rewrite its final term to an equivalent fraction by multiplying it by $a/a$. | $4\left(x^2 - \frac{8}{4}x - \frac{5}{4} \bullet \frac{4}{4}\right)$ $= 4\left(x^2 - \frac{8}{4}x - \frac{20}{16}\right)$ |
| 4 | Having accounted for the denominators, we need only to consider the numerators. Find two numbers whose sum is $b$ and whose product is $a \cdot c$. | For this, we need two numbers whose sum is –8 and product is –20 since the sum of fourths is still fourths but the product of fourths is sixteenths. Our numbers are 2 and –10. |
| 5 | Rewrite in factored form, with denominators of a for the two numbers found in step 4. Also, simplify fractions where possible. | $4(x + 2/4)(x - 10/4)$ <br> $4(x + 1/2)(x - 5/2)$ |
| 6 | *Optional: If desired, distribute your leading coefficient to eliminate fractions in the factored form. | $2 \cdot 2(x + 1/2)(x - 5/2)$ <br> $2(x + 1/2) \cdot 2(x - 5/2)$ <br> $(2x + 1)(2x - 5)$ |

## EXPANDING OLD KNOWLEDGE TO CREATE NEW KNOWLEDGE

Factoring for roots (see the steps in **table 2**) allows students to use the same strategy as when factoring $x^2 + bx + c$. This reutilization of prior knowledge reinforces the concept that mathematics is not a set of disconnected topics but rather interconnected ideas reinforced by one another. Students find it especially illuminating when we discuss how this new strategy directly transforms to the old strategy in the case when $a \cdot c = c$ since $a = 1$.

A tangential benefit is that the new method allows students an opportunity to practice and refine arithmetic with fractions. This is important because, according to Siegler and his colleagues (2012), the ability to achieve competency in fraction multiplication and division is the number one predictor of high school mathematics achievement, even after controlling for factors such as overall intellectual ability, working memory, and socioeconomic status.

One might argue that Siegler and his colleagues predict achievement of high school students—and my students are already in high school. Those students who arrive not completely proficient in fraction arithmetic might make some serious gains by employing the factoring for roots strategy. Using fraction arithmetic as a fundamental tool to arrive at a more cognitively demanding mathematical abstraction may help them. Cognitive scientist Willingham states that students are more likely to remember concepts used to arrive at novel and more complex ideas than if the concept itself is the final learning outcome (2009).

Finally, factoring for roots does not leave itself vulnerable to a student using it without fully comprehending the mathematics behind the technique.

**It ties procedural fluency and conceptual understanding together in such a way that students find the factoring for roots method useful and enlightening.**

This is because factoring for roots requires correctly manipulating fractions. In this manner, it ties procedural fluency and conceptual understanding together in such a way that students find the factoring for roots method useful and enlightening. Compared with other techniques, factoring for roots is superior in three ways:

1. It builds on previous knowledge.
2. It is conceptually based.
3. It avoids case-by-case analysis.

Instead of mathematics being a disconnected set of procedures, this method attempts to combine two ideas that come from the same concept, factoring, whereas other textbooks and strategies push them further apart.

## REFERENCES


Brownell, William. 1935. "Psychological Considerations in the Learning and Teaching of Arithmetic." In *The Teaching of Arithmetic, 10th Yearbook of the National Council of Teachers of Mathematics*, edited by W. D. Reeve, pp. 1–31. New York: Bureau of Publications, Teachers College Columbia University.

Carter, John A., Gilbert J. Cuevas, Roger Day, and Carol Malloy. 2010. *Algebra 1*. New York: Glencoe/McGraw-Hill.

Hiebert, James. 1999. "Relationships between Research and the NCTM Standards." *Journal for Research in Mathematics Education* 30, no. 1 (January): 3–19.

Larson, Ron, and Laurie Boswell. 2015. *Big Ideas Math: A Common Core Curriculum*. Erie, PA: Big Ideas Learning.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

———. 2014. *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: NCTM.

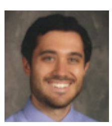
———. "Procedural Fluency in Mathematics." Position paper. National Council of Teachers of Mathematics. Accessed January 1, 2017. http://www.nctm.org/Standards-and-Positions/Position-Statements/Procedural-Fluency-in-Mathematics/

National Research Council (NRC). 2001. *Adding It Up: Helping Children Learn Mathematics,* edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell. Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academies Press.

Siegler, Robert S., Greg J. Duncan, P. E. Davis-Kean, Kathryn Duckworth, Amy Claessens, Mimi Engel, Maria I. Susperreguy, and Meichu Chen. 2012. "Early Predictors of High School Mathematics Achievement." *Psychological Science* 23, no. 7 (July): 691-97. doi:10.1177/0956797612440101

Van De Walle, John A, LouAnn H. Lovin, Karen S. Karp, and Jennifer M. Bay-Williams. 2014. *Teaching Student-Centered Mathematics*. Boston: Pearson, 2014.

Willingham, Daniel T. 2009. *Why Don't Students Like School? A Cognitive Scientist Answers Questions about How the Mind Works and What It Means for Your Classroom*. 1st ed. San Francisco, CA: Jossey-Bass.


**ADAM CLINCH,** aclinch@helenaschools.org, teaches algebra 1, honors 1, college algebra, and statistics at Capital High School in Helena, Montana. He is interested in incorporating the history of mathematics and displaying the interconnections between mathematical ideas across different domains in his classes.