Honors Precalculus Name: ___________________________________

Chapter 7 Notes

7.1/7.2 Solving Systems of Two Equations Review

There are three tools to solve a system: 1) Graphing 2) Substitution 3) Elimination

Ex: Solve the following system by substitution. $y = \frac{1}{2}x - 6$ $2x + 4y = 5$	Steps for substitution 1) Solve for one of the variables 2) Substitute into the other equation and solve 3) Substitute the value you obtained from step 2 into EITHER of the original equations and solve
A total of $12,000 is invested in two funds paying 5% (high risk) and 3% (low risk) simple interest a year. If the investor earns $500 in interest in the year, how much was invested in each?	
Ex: Solve by elimination $5x - 6y = -7$ $-3x - 4y = 27$	Steps for Elimination 1) Multiply one (or both) equation so the coefficients match for either x or y 2) Add or subtract the equations 3) Solve for the value of the non-eliminated variable 4) Substitute the value you obtained from step 3 into EITHER of the original equations and solve

Solving Unique Systems of Equations

$15x - 5y = 10$ $y = 3x - 2$	$2x + 3y = 8$ $4x + 6y = 12$

# of Solutions	Graphically	Algebraically
1		
0		
Infinite		

Break-Even Point: When $R(x) =$ ________

Profit Function: $P(x) =$

Ex: A company that manufactures bicycles has a fixed cost of $100,000. It costs $100 to produce each bicycle and they sell them for $350.

a) Find the break-even point. (First define your variable)

b) Define your profit function and estimate their profit if they sell on average 3 bikes per workday for 2 years.

7.3 Solving a System of Equations in Three Variables Through Gaussian Elimination

Row-Echelon Form: a system of equations where the last equation is solved for and you can use back-substitution to solve for the remaining variables. The solution (x, y, z) is called an ordered triple.

Ex: Use back substitution to solve for the system that is in row-echelon form.

$$\begin{aligned} 2x - y + 5z &= 22 \\ y + 3z &= 6 \\ z &= 3 \end{aligned}$$

Gaussian Elimination: process of converting a system of equations into row-echelon form. There are three allowable operations: 1) ________________ two equations 2) _________________ one of the equations by a nonzero constant and 3) _______ or ___________________ two equations.

Gaussian Elimination with One Solution. Steps:

1) Eliminate x from middle using R1 and R2 2) Eliminate x from last using R1 and R3 3) Eliminate y from last using R2 and R3

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ 3x + y - z &= 2 \end{aligned}$$

Gaussian Elimination with Unique Solution

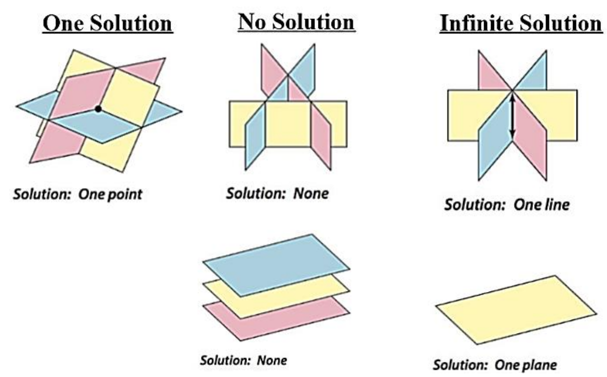
$$\begin{aligned} x - 3y + z &= 1 \\ 2x - y - 2z &= 2 \\ x + 2y - 3z &= -1 \end{aligned}$$

Gaussian Elimination with Unique Solution

$$x + y - 3z = -1$$
$$y - z = 0$$
$$-x + 2y = 1$$

Possibilities When Solving a System

# of Solutions	Graphically	Algebraically
1	Cross at one point	x = #, y = #, z = #
0	Don't all cross at one point	$0 \neq 2$
Infinite	Cross along a line or all same plane	$2 = 2$

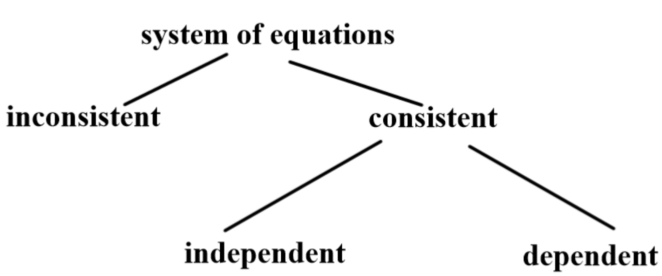


Other vocabulary for the number of solutions in a system:

Inconsistent: a system is inconsistent when there are no solutions

Consistent: a system is consistent when there is at least one solution. There are two types of consistent systems:

1. **independent**: a consistent system with just one solution
2. **dependent**: a consistent system with infinite solutions



7.4 Matrices, Gauss Jordan Elimination and RREF

Matrix: rectangular array of numbers. An m x n matrix has m rows and n columns.

$$\begin{array}{cc} & \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \cdots & \text{Column } n \end{array} \\ \begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Ex: State the dimension of each matrix below.

$[5]$	$\begin{bmatrix} 1 & -6 & 3 \end{bmatrix}$	$\begin{bmatrix} 8 & -7 \\ 3 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 4 \\ -3 & 0 \end{bmatrix}$	$\begin{bmatrix} 11 \\ -17 \end{bmatrix}$

Ex: Write the following system as an augmented matrix and then solve using Gaussian Elimination and back substitution.

$$\begin{aligned} x + 3y + 4z &= 7 \\ 2x + 7y + 5z &= 10 \\ 3x + 10y + 4z &= 27 \end{aligned}$$

Reduced Row-Echelon Form: a matrix where every column that has a leading 1 has zeros in all positions above and below the 1 (you can read off the answer).

$$\left[\begin{array}{ccc|c} 2 & -1 & 5 & 22 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Row-Echelon Form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Reduced Row-Echelon Form

Ex: Take the problem we got to Row-Echelon Form earlier and get it to Reduced Row-Echelon Form.

Ex: Solve the following system using the RREF function on a calculator

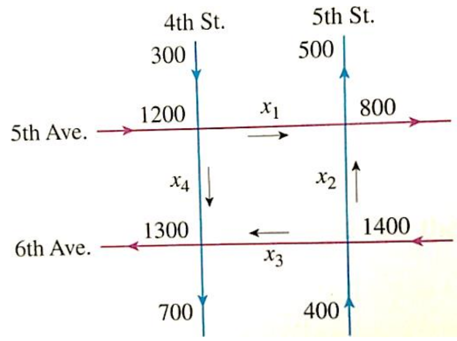
1) Go to the matrix menu 2) Enter your matrix 3) Quit the matrix editor 4) Go to matrix menu and select "rref"
5) Select your matrix and hit Enter

$$x + 3y + 4z = 7$$
$$2x + 7y + 5z = 10$$
$$3x + 10y + 4z = 27$$

Application of Matrices and RREF

The figure shows the flow of traffic near a city's downtown during rush hour on a typical weekday. 5th and 6th Ave. can have a maximum of 2000 cars/hour without causing congestion, whereas 4th and 5th St can only have 1000. The flow of traffic is controlled by traffic lights at each of the five intersections. *To avoid congestion, assume all traffic entering an intersection must leave it.

Suggest a possible traffic flow pattern that ensure no traffic congestion.



7.5 Day 1: Matrix Operation

Matrix Addition: Matrices with the same dimension can be added by adding matching elements.

$$\begin{bmatrix} 1 & -6 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 7 \\ 5 & 5 \end{bmatrix} =$$

Scalar Multiplication: Every element in the matrix is multiplied by the scalar.

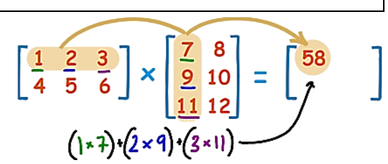
$$-2\begin{bmatrix} 1 & -6 \\ 4 & 3 \end{bmatrix} =$$

Let A and B be the matrices defined. Then find the following. $A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$ $B = \begin{bmatrix} -4 & -8 \\ -2 & 0 \\ 14 & -6 \end{bmatrix}$

$A - B$	$2A + 2B$	$2(A + B)$

Solve for matrix X in the matrix equation $2X - 2A = B$

Matrix Multiplication: a row-by-column multiplication as shown. If A is an *m x n* matrix and B a *n x p* matrix, then the product, AB, will be an *m x p* matrix.



Let A, B, & C be the matrices defined. Then find the following. $A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

AB	BC
	Note: C is known as the identity matrix for 2x2 matrices

State whether each is possible. If so, do it.

$$\begin{bmatrix} 3 & -8 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -4 \\ 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -6 & 1 \end{bmatrix}$$

Find the total cost for each team's supplies.

Cost for each	Bats	Balls	Gloves
	\$90	\$6	\$60

Equipment	Women's team	Men's team
Bats	12	15
Balls	45	38
Gloves	15	17

7.5 Day 2 – Matrix Operations with a Calculator and Applications

Let A, B, and C be the matrices defined. Then use a calculator to find the following.

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -7 & 5 \\ 4 & 9 \end{bmatrix}$$

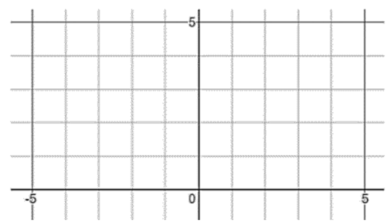
$A(B+C)$	$AB + AC$	$(B+C)A$

As you can see, the distributive property applies, but it is not commutative. Any guesses what the distributed form of (B + C)A looks like?

Ex: Use a calculator to show the following results.

$(A+B)^2 \neq A^2 + 2AB + B^2$	$(A+B)^2 = A^2 + AB + BA + B^2$

Application 1 of Matrix Multiplication



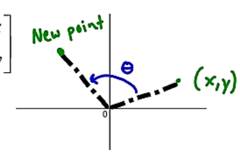
1) Plot the point (4,1) on the graph.

2) Let the point (4,1) be written in a matrix as $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

3) Take the matrix form of the point (4,1) and have it multiply the following matrix. Plot the result on the graph and see what you notice. $\begin{bmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} =$	4) Now do the following, plot the result, and see. $\begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} =$

Rotation Matrix: If you want to rotate a point (x,y) in the coordinate plane by an angle θ, you perform the following matrix computation.

$$\underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\text{This is called the rotation matrix.}}\begin{bmatrix} x \\ y \end{bmatrix}$$



Application 2 of Matrix Multiplication: A real estate firm builds houses in three states. The projected number of units of each model, for each state, is given by the matrix.

Model

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \\ \end{array}$$

$$A = \begin{bmatrix} 60 & 80 & 120 & 40 \\ 20 & 30 & 60 & 10 \\ 10 & 15 & 30 & 5 \end{bmatrix}\begin{array}{l} \text{NY} \\ \text{Conn} \\ \text{Mass} \end{array}$$

The profits to be realized for each model, respectively, are \$20k, \$22k, \$25k, and \$30k. Define a second matrix and use a calculator to determine the total profit to be realized by the firm, in each state.

7.6 Day 1 Define and Derive the Inverse Matrix

Verify that if matrix A is given below, B is the inverse of A (denoted A^{-1}, once we verify it is the inverse).

$A = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2.5 \\ -1 & 3 \end{bmatrix}$

Inverse Matrix: A^{-1} is the inverse matrix of matrix A if $A \cdot A^{-1} = I = A^{-1} \cdot A$ where I is the identity matrix.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

provided $ad - bc \neq 0$.

Proof for Deriving the Inverse Matrix Equation

Let A be the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want to find the inverse matrix such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Through matrix multiplication we obtain:

Writing this out as a system of equations we get the following.

$$
\begin{array}{llllr}
(1) & ax & & +bz & =1 \\
(2) & & ay & & +bw=0 \\
(3) & cx & & +dz & =0 \\
(4) & & cy & & +dw=1
\end{array}
$$

Let's find x first. To solve for x, we need to get rid of z in equations (1) and (3). To do this, multiply equation (1) by d and equation (3) by b and then subtract the equations.

Now, let's find y. To solve for y, we need to get rid of w in equations (2) and (4). To do this, multiply equation (2) by d and equation (4) by b and then subtract the equations.

Now, let's find z. To solve for z, we need to get rid of x in equations (1) and (3). To do this, multiply equation (1) by c and equation (3) by a and then subtract the equations.

Now, you find w.

So, we found the inverse matrix A^{-1} is

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

Dividing the common factor out of each entry we arrive at the desired result.

$A^{-1} =$

Ex: Use the formula we derived to show that in the first example, B was the inverse of A.

$A = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2.5 \\ -1 & 3 \end{bmatrix}$

Ex: Use the formula we derived to show that the matrix A below will not have an inverse.

$A = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

7.6 Day 2 and 7.7 Use Inverse Matrices to Solve and 2x2 Determinants

Solving a Matrix Equation: If A is an invertible matrix, the solution for AX = B will be $X =$______________

Ex: Use an inverse matrix and solve the following system by hand.

$3x + 4y = -2$
$5x + 3y = 4$

1) Write as a matrix equation.

2) Find the inverse of the coefficient matrix.

3) Multiply both sides by the inverse matrix and simplify.

Use a calculator and solve using an inverse matrix.

$3x + 4y = -2$ $5x + 3y = 4$	$x + y + z = 0$ $3x + 5y + 4z = 5$ $3x + 6y + 5z = 2$
Consider a $500,000 investment option consisting of AAA-rated bonds, A-rated bonds, and B-rated bonds. The average yield for each are 4.5% on AAA, 5% on A, and 7% on B, with an annual total yield of $28,000. This package involves twice as much B rated bonds as A-rated bonds. Let x = $ in AAA, y = $ in A, and z = $ in B. Set up a system of three equations and solve using an inverse matrix with a calculator.	

room on next page, if needed

Determinant of a 2x2 Matrix: For a matrix A, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant, denoted det(A) or |A| = ad – bc

If a det(A) = _____ then the matrix A has no inverse.

If a matrix has no inverse and it is the coefficient matrix in a system of equations, that system will have _____ solutions.

Ex: Determine whether each matrix has an inverse or not and state why based on the determinant.

$\begin{bmatrix} -5 & 2 \\ 6 & 3 \end{bmatrix}$	$\begin{bmatrix} 2 & -4 \\ 6 & -12 \end{bmatrix}$

Ex: Use the determinant to show why this system has no solutions.

$$-12x + 24y = 7$$
$$2x - 4y = 5$$