Honors Precalculus Name: ______________________________________

Chapter 3 Notes

3.1 Day 1 - Exponential Functions, Their Graphs, and Transformations

Exponential Function**:** Function of the form $f(t) = ab^t$ where a is the ________________________and b is the_________________

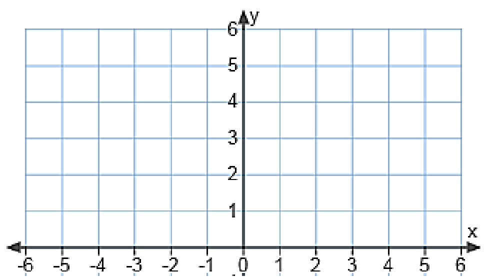
Example: Write the equation for each exponential function.

x	-2	-1	0	1	2
y	0.75	1.5	3	6	12

x	-2	-1	0	1	2
y	81	27	9	3	1

Example: Graph the function $y = 2^x$ and identify the domain and range. Note: $y = 2^x = 1(2^x)$

x	y
-2	
-1	
0	
1	
2	

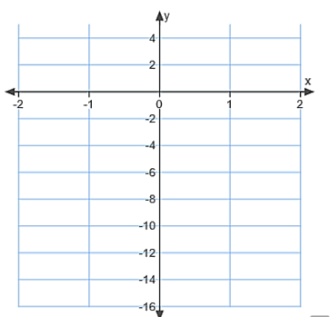


Domain:

Range:

Example: Graph the function $y = -4(0.5)^x$ and identify the domain and range.

x	y
-2	
-1	
0	
1	
2	



Domain:

Range:

Growth:

Decay:

Transformations of Exponentials: Explain how each transform the graph $f(x) = 2^x$.

$g(x) = 2^{x-1}$	
$g(x) = 2^{x+1}$	
$g(x) = 2^x - 1$	
$g(x) = 2^x + 1$	

$g(x) = -2^x$	
$g(x) = 2^{-x}$	
$g(x) = 4(2)^x$	
$g(x) = 1/4\ (2)^x$	

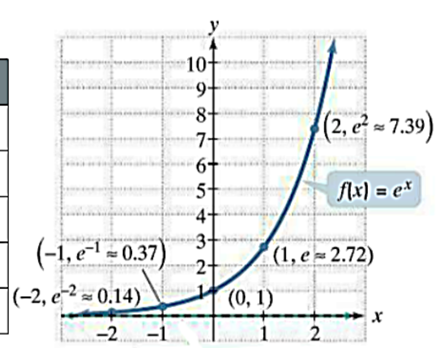
Example: Use the graph and table for $f(x) = e^x$ to make a graph of $g(x) = e^{x-1} + 2$.

$f(x) = e^x$

X	Y
-2	0.14
-1	0.37
0	1
1	2.718
2	7.39

$g(x) = e^{x-1} + 2$

X	Y



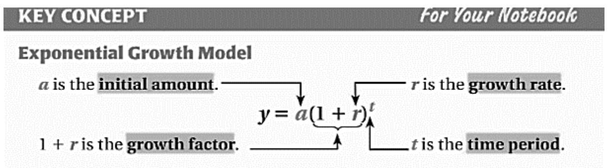
3.1 Day 2 - Compound Interest, Graph $y=ae^{bx}$, Applications

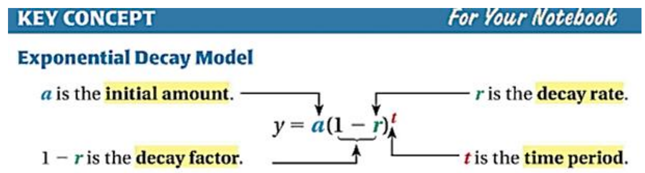
Modeling Population Growth

The population of Helena in the year 2000 was 23,000. The U.S. Census Bureau found that the population then increased, on average, by 2% each year. What was the population in 2005?

Year	0	1	2	3	4	5
Population						

Example: An investor places $250,000 in an account that earns 4% interest each year. How much will it be worth in 5 years?





Compounding Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A = amount after t time

P = principal (original amount)

r = rate of growth (as decimal)

n = number times compound annually

t = length of time

Example: The average student loan debt is $36,000. If you took out $9000 your freshman year and didn't make a payment for four years, how much would that $9000 loan grow to at 7% annual interested compounded:

a) quarterly?

b) monthly?

Lead-In: Introducing Compounding Continuously

Suppose you invest one whole dollar in an account that pays 100% interest compounded n-times per year. Find the balance for each frequency after one year.

Balance Equation =

Frequency	n value	Balance After 1 Year
annually	1	$1.00\left(1+\frac{1}{1}\right)^{1} =$
quarterly	4	$1.00\left(1+\frac{1}{4}\right)^{4} =$
monthly	12	$1.00\left(1+\frac{1}{12}\right)^{12} =$
daily	365	$1.00\left(1+\frac{1}{365}\right)^{365} =$
hourly	8,760	$1.00\left(1+\frac{1}{8760}\right)^{8760} =$
each second	31,536,000	$1.00\left(1+\frac{1}{31{,}536{,}000}\right)^{31{,}536{,}000} =$

Formulas for Compound Interest

After t years, the balance, A, in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas:

1. For n compounding periods per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$.

Example Revisited: The average student loan debt is $36,000. If you took out $9000 your freshman year and didn't make a payment for four years, how much would that $9000 loan grow to at 7% annual interested compounded continuously?

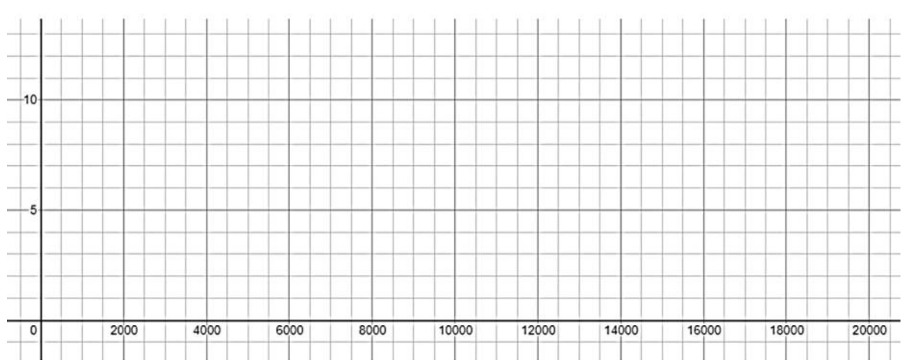
Ex: Carbon-14 is a radioactive isotope of carbon with an atomic nucleus containing 6 protons and 8 neutrons. Its presence in organic materials is the basis a type of radiocarbon dating method.

For a sample that starts with 10 grams, the quantity of carbon-14 remaining after t years can be found using

$$Q(t) = 10\left(\frac{1}{2}\right)^{\frac{t}{5700}}$$

Use a calculator to make a table and graph for $0 < t < 20{,}000$

t	0	4000	8000	12,000	16,000	20,000
$Q(t)$						



3.2 Day 1 - Definition of Logarithms and Basic Properties

Warm-Up: A house purchased in 1990 for $125,000 has increased in value by 3% each year since.

a) What would its value be today (assume compound annually unless otherwise noted)?	b) When will it reach a value of $400,000?

Lead-In: Last section we would set up exponential function like Q = 10^t . We were only ever able to find Q if given a value for t, but how can we find t if given Q? For example: $10^t = 2500$.

$$10^2 = 100, \qquad 10^3 = 1000, \qquad 10^4 = 10{,}000$$

Common Logarithm (Log) Function: If x is a positive number, log(x) equals the exponent of 10 such that $10^{exponent} = x$.

Mathematically: if $\log(x) = y$ then ______________ or If $10^y = x$ then ______________

log(100) = power of 10 that gives 100 =

log (100) = 2 since $10^2 = 100$ log(2500) = since

log(0.01) = since log(-10) = since

Natural Logarithm (Ln) Function: If x is a positive number, ln(x) equals the exponent of e such that $e^{exponent} = x$.

Mathematically: if $\ln(x) = y$ then ______________ or If $e^y = x$ then ______________

ln(e) = ____ since

General Log Function: $log_b(\ )$: If x is a positive number, $\log_b(x)$ equals the exponent of *b* such that $b^{exponent} = x$.

Mathematically: if y = $\log_b(x)$ then ______________ or if x = b^y then ______________

$\log_7(49)$ = since $\log_3(81)$ = since

$\log_{36}(6)$ = since $\log_5(1)$ = since

Example: If written in log form, write as an exponential. If written as an exponential, write in log form.

$4 = \log(10{,}000)$ ⟶	⟶ $5^4 = 625$
$6 = \log_2(64)$ ⟶	⟶ $e^3 = 20.0855$

Basic Logarithm Properties

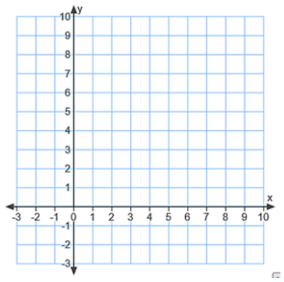
General Properties	Common Logarithms	Natural Logarithms
1. $\log_b 1 =$	1. $\log 1 =$	1. $\ln 1 =$
2. $\log_b b =$	2. $\log 10 =$	2. $\ln e =$
3. $\log_b b^x =$	3. $\log 10^x =$	3. $\ln e^x =$
4. $b^{\log_b x} =$	4. $10^{\log x} =$	4. $e^{\ln x} =$

(Properties 3 and 4: Inverse Properties)

3.2 Day 2 – Graphs of Logarithms and Transformations

Graph of Common Logarithm (Log)

X	log(x)
0.001	
0.01	
0.1	
1	
10	



Domain:

Range:

Other Graphs of Log Functions: Note, all logarithmic functions look the same in general.

For the graph of *f(x) = log$_b$(x)* , the graph ___________________ as you increase the value of b (for b > 1).

Characteristics of General Log Graphs *y = log$_b$(x)*

1. The domain is ______________.   The range is ________________________.

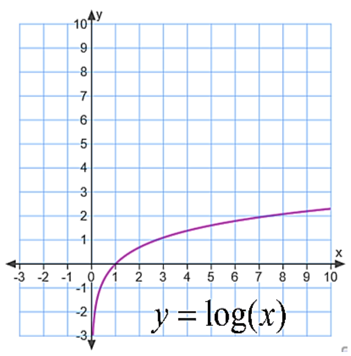
2. They all pass through the point (1,   ).

3. They have ____ y -intercepts because there is a vertical asymptote at _____________.

Logarithms and Exponents are Inverses:

On the coordinate plane you used prior, also graph y = 10^x.

X	10^x
-2	
-1	
0	
1	
2	

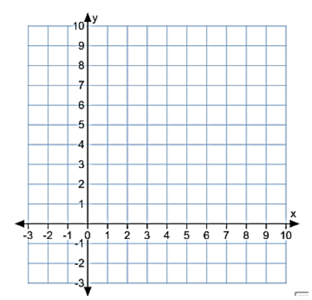


The equation *f(x) = log$_e$(x)* has a special notation because it is used so often in math. It is denoted *f(x) = ln(x)* and is called the **natural log** function.

If *g(x)* = e^x, fill in the table for *g(x)* and use it to fill in *f(x)*'s table

X	$g(x) = e^x$
-2	
-1	
0	
1	
2	

X	f(x) = ln(x)



Transformations of Log Graph: Explain how each transform the graph $f(x) = \log(x)$.

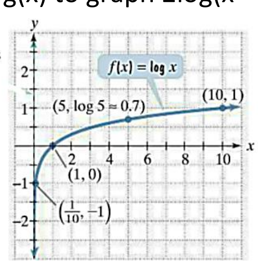
$g(x) = \log(x) + c$	
$g(x) = \log(x) - c$	
$g(x) = \log(x + c)$	
$g(x) = \log(x - c)$	

$g(x) = -\log(x)$	
$g(x) = \log(-x)$	
$g(x) = c \cdot \log(x)$	
$g(x) = \log(cx)$	

Example: Use the graph of log(x) to graph 2log(x - 1) - 3

X	log(x)
0.01	-2
0.1	-1
1	0
10	1
100	

X	2log(x - 1) - 3



Ex: The magnitude, R, on the Richter scale of an earthquake of intensity I is given by $R = \log(I/I_0)$ where I_0 is the intensity of a nearly negligible zero-level quake. The earthquake in Helena during the fall of 1935 was 1,585,000 times as intense as a zero-level. What was it on the Richter scale?

3.3 – Properties of Logarithms

Log Properties

Example	**Property Name**	**Property**
$\log(10^2 \cdot 10^4) =$		
$\log\left(\frac{10^8}{10^6}\right) =$		
$\log((10^2)^4) =$		

Example: Simplify as much as possible using the log properties.

$\log(100x)$	$\log\,(10x^2)$
$log_2(\frac{x}{8})$	$\ln\,(\sqrt{e})$

Properties for Expanding Logarithmic Expressions

For $M > 0$ and $N > 0$:

1. $\log_b(MN) = \log_b M + \log_b N$ Product rule
2. $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$ Quotient rule
3. $\log_b M^p = p\log_b M$ Power rule

Properties for Condensing Logarithmic Expressions

For $M > 0$ and $N > 0$:

1. $\log_b M + \log_b N = \log_b(MN)$ Product rule
2. $\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$ Quotient rule
3. $p\log_b M = \log_b M^p$ Power rule

Example: Use the log properties to expand as much as possible.

$\log\,(x^5 \cdot \sqrt[3]{y})$	$\log\left(\frac{\sqrt{x}}{100y^4}\right)$

Example: Use the log properties to write as a single logarithm.

$2\ln(\text{x}) + \frac{1}{3}\ln\,(x+5)$	$2\log(x-3) - \log\,(x)$

3.4 Day 1– Solving Exponential Equations

Lead-In: Prior, if we had an exponential equation like $A = 5000(1.03)^t$, we would not be able to solve for *t* to find the time it takes for the account to grow to \$6000. We will see how to do that today.

Begin by simplifying: $\log(x^2) =$ $\log(4)^x =$. What did the log do to the power?

Using Logarithms to Solve Exponential Equations

1. Isolate the exponential expression

2. Take the log of both sides (almost always the common logarithm unless you are working with powers of *e*)

3. Simplify the expression and solve.

$3^x + 4 = 12$	$2 \cdot 10^x - 12 = 80$	$4e^{2x} + 6 = 86$ *by natural log	$4e^{2x} + 6 = 86$ *by common log

Solving With Variable in Exponents on Both Sides or Different Bases

Why is this "easy" to solve? $5^{3x+4} = 5^{x-6}$

Solve by matching bases $27^{x+3} = 9^{x-1}$	Solve by using logs $27^{x+3} = 9^{x-1}$

Solve Unique Scenarios and Applications

Ex: Solve the following: $e^{2x} - 3e^x + 2 = 0$	**Lead-In:** You invest \$5000 in a bond returning 3% compounded annually. How long will it take for the investment to grow to \$6000?

3.4 Day 2– Solving Logarithmic Equations

Recall, $\log(100) =$, since . So, if $\log(x) = a$, we know that

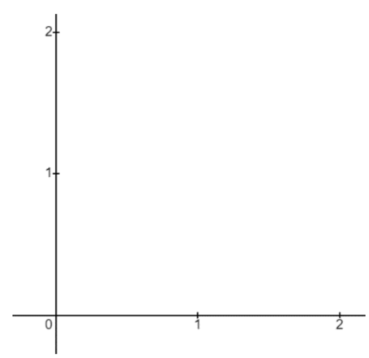
Using Logarithm Definition to Solve Logarithm Equations

1. Isolate the logarithm expression

2. Use the definition of the logarithm to rewrite the equation in exponential form.

3. Solve and check.

$log_2(x-4)+7=10$	$4\ln(3x)=8$	$log_2(x)+\log_2(x-7)=3$

$\ln(x-2)+\ln(2x-3)=2\ln(x)$	Some equations cannot be solved numerically (using algebra) and rely on a computer. Consider the following equation. $\ln(x) = x^2 - 2$ 

Aside: Prove the change of base formula.

Say $log_b(a) = n$ then $b^n = a$. Solve for n.

3.5 Day 1 – Solving Exponential Growth, Decay and Log Word Problems

Exponential Growth/Decay Equation: $A = A_0 e^{kt}$ where A_0 = original amount, k = growth/decay rate, t = time

For the following, it is growth when ____________________ and decay when ___________________

Yearly Growth	Compounding Interest	Compounding Continuously
$y = a(1+r)^t$ -Used to track the value of items over time. Ex: The value of a $160 thousand house increases by 4% annually. Eq:	$y = a\left(1+\frac{r}{n}\right)^{nt}$ -Used exclusively for banking purposes. Ex: A $700 computer is purchased on a credit card at 18% interest compounded monthly. Eq:	$y = ae^{rt}$ -Used to model real-life data. -A population of wolves in a national park starts out at 110 and increases by 20% per year. Eq:

Ex: The number of total confirmed cases of COVID-19 worldwide during the spring/summer of 2020 followed an exponential model. Use the two points from two different days to determine the exponential equation for this if we let t be days since March 1.

Step 1: find the exponential growth rate, *k*

Step 2: find the initial amount

Ex: The equation we should have obtained was __________________.

1) What do each of the values in the equation represent in real-life?	2) Use your equation to estimate the total number of confirmed cases worldwide by Sept 1 (184 days).

<u>Example</u>: Assume there is a savings account that provides interest compounded continuously at interest rate, 3%. How long would it take to double?

<u>Carbon-14 Dating</u>: used to date fossils or artifacts by using the fact that carbon-14 has a half-life of________ years.

<u>Half-life</u>: amount of time it takes something to decay to __________________ of the original amount.

<u>Example</u>: In 1991, the body of Otzi was found by two hikers in the Alps of Northern Italy. Examinations revealed that his tissue contained 47% of its original carbon-14. When did he die?

<u>Step 1:</u> use the half-life to find decay rate *r*.

<u>Step 2:</u> use r and the entire equation to find age.

<u>Example</u>: An isotope has a half-life of t_h. Find a general formula for the decay rate of the isotope if you are given its half-life.

3.5 Day 2 – Log and Logistic Model

Example ph Application: The pH-scale measures the acidity of different solutions by using a log scale to record the concentration of hydrogen ions. The equation is **pH = -log(x)** with x = hydrogen ion concentration in moles/liter

Question: *Find the hydrogen ion concentration for the most acidic rainfall ever with a pH of 2.4.*

Ex: The following equation relates the level of sound β (in decibels) with an intensity *I* where I_0 is the faintest sound audible to the human ear at 10^{-12} Watts/m^2. Equation: $\beta = 10\log\,(I/I_0)$.

a) Find the dB reading for $I = 10^{-4}\ W/m^2$ (door slamming)	b) Find the dB reading for $I = 0.1\ W/m^2$ (Grizzly football game)
c) Through installing noise suppression material in an auditorium, the noise level decreased from 93 dB to 80 dB. What was the percent decrease in the intensity level, I?	

Introducing the Logistic Growth Model: In a prior lesson, we modeled the spread of COVID-19 across the world and found the equation for the number of total cases to be $A = 0.389e^{0.0248t}$ where A = cases (millions), t = days since March 1, 2020.

What is the problem with this model? What does it predict will always be happening? Is that a fair assumption?

Logistic Growth Model

The mathematical model for limited logistic growth is given by

$$f(t) = \frac{c}{1 + ae^{-bt}} \quad \text{or} \quad A = \frac{c}{1 + ae^{-bt}},$$

where a, b, and c are constants, with $c > 0$ and $b > 0$.

As time increases ($t \to \infty$), the expression ae^{-bt} in the model approaches 0, and A gets closer and closer to c. This means that $y = c$ is a horizontal asymptote for the graph of the function. Thus, the value of A can never exceed c and c represents the limiting size that A can attain.

Ex: The function below describes the number of people, f(t), who have become ill with a virus t weeks after its initial outbreak in a town with 30,000 inhabitants. *Assuming no preventative measures are taken

$$f(t) = \frac{30,000}{1 + 1200e^{-0.5t}}$$

a) How many people were sick when the epidemic began?

b) How many people were sick after 2 weeks?

c) What is the limiting size of the population that can become sick?

d) If the town will lockdown when 5% of the people become sick, how long is that?

Sketch of the graph of the function