Increasing Student Conceptual Understanding in a College Algebra Course

Through Multiple Cognitive Practices

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Introduction

It is no secret that in introductory 100-level math courses at universities and colleges across the United States, there is a heavy focus on teaching students, in a short period of time, how to perform mathematical operations. For example, in a College Algebra course that focuses on the study of functions, students are expected to discover how to solve exponential, logarithmic and polynomial expressions within three weeks of each other. The curriculum consists of 180 days of high school material condensed into a 45-day university schedule. Due to this rapid-fire method of teaching, where each day a new topic is introduced and very little time is available for digestion, conceptual understanding can suffer. Because of this trend at the K-12 level, we have seen the new Common Core State Standards for Mathematics adopt a “Toward a Greater Focus and Coherence” policy that emphasizes conceptual understanding (Mathematics: Toward a Greater Focus and Coherence, 2012).

At the University of Montana, I am an instructor for Math 121 – College Algebra, a course that is meant to strengthen students’ algebra skills before they enroll in trigonometry or pre-calculus. The course is focused on the study of functions and their inverses, which include linear, quadratic, polynomial, rational, exponential, and logarithmic functions. During my time as an instructor, I have seen a recurring issue during class discussions, assessments, office hour meetings, and email dialogues, in which students focus heavily on the method of solving a problem without regard to why certain techniques are being applied. They are extremely proficient at repeating a procedure I have demonstrated but the moment the question becomes flipped, they freeze. This is not an unusual phenomenon in a standards-based world or where class pace is best described as frantic, especially since humans are not naturally good at applying or generalizing (Willingham, 2009). In a class period where there is much material to discuss and
a small amount of time to do it, my question became how could I increase the conceptual understanding of my students within those constraints?

As the focus of my question shifted towards the study of cognition, I came across a number of ways in which students can become better conceptual learners. The first is to increase their background knowledge. A problem that exists in most classrooms is that students have varying levels of background knowledge and people are more easily able to retain new information if they already know a considerable amount about the domain (Alexander, Kulikowich, & Shulze, 1994).

Secondly, the best way to increase the odds that a person remembers is to have them actively think about the item, concept, or lesson. It is not enough that students understand in the moment the process of how to find the exponential equation containing two points, rather cognitive scientists are now stating students must consciously think about the process and why they are doing what they’re doing in order for the memory to become concrete (Schacter, 2002).

Finally, in order to help students transfer abstract knowledge to new concepts we must practice directly comparing situations. The best way to do this is through a group or class discussion, compare two problems that differ in their surface structure, perhaps one is about the size of a front yard and the other concerns the amount of paint for a wall, but have similar deep structures, in that their solution steps are the same. These experiences help students develop their problem-solving techniques and aids in conceptual understanding (Willingham, 2009).

My questions then became how can I increase my students’ background knowledge, encourage them to consciously think about what they are learning and implement comparative strategies in my classroom? To answer these I thought back on my own education and experiences to develop strategies to answer my questions.
During the spring of 2013, I took Abstract Algebra II, a course that focuses on a deeper investigation of groups, rings, fields, vector spaces and field extensions and I had an instructor who focused heavily on knowing the vocabulary. Other math teachers of mine simply usually wrote a new word’s definition down once and moved on to the next item. In this professor’s teaching, he consistently brought up what it meant to be specific terms and what consequences came with those terms. On tests and quizzes, a fourth of the assessment was simply defining words, a rare occurrence in upper-level math classes. However, I quickly realized that knowing the terms and their properties made solving the more difficult abstract problems much easier. With that experience as motivation, my question for this became, will putting a greater focus on vocabulary increase my students’ background knowledge and in turn increase their conceptual understanding?

In my Content Literacy course for my M.Ed., my teacher required us to keep a journal to reflect upon our experiences as teaching assistants or course instructors. We read articles about being reflective teachers and how teachers should take the time to analyze what they are doing in the classroom. I found by setting aside time, organizing my thoughts and putting them to paper, it was much easier to realize what was and wasn’t working in my teaching. I then thought, why don’t we ask students to do this? My research question for this topic has become, does having students reflect on the day’s lesson allow them to make deeper connections and improve their conceptual understanding?

Finally, throughout my career as a math education graduate student I have come across the works, theories, and principles of mathematics education specialist Zoltan Dienes numerous times. A specialist in learning theory, one of his principles is known as the multiple embodiment principle and states, “For the child to abstract mathematical concepts fully, the concepts should
be presented in multiple embodiments that are as visually different as possible,” (English, 1995).

Dienes laid the groundwork for what current cognitive scientists call surface and deep structure. My inquiry in this regard is does presenting group discussions in which the class as a whole directly compares two examples, under the assumptions of the multiple embodiment principle, increase students’ conceptual understanding?

My research focuses on three main areas, the role of vocabulary, reflection, and multiple embodiments in the classroom. Instead of concentrating on just one, I have chosen to study their combined effects, as practicing multiple cognitive exercises in the classroom helps students see the varying ways items can be processed, (Willingham, 2009). As I formalized my questions and goals, I finally arrived at the following set.

**Research Questions**

- By emphasizing the mathematical vocabulary in my classroom, will students become more attentive to understanding what mathematical terms mean and imply?
  - Will this increase their background knowledge?
  - Will it make solving problems easier?

- Through the use of reflection, will my students retain the lesson better and be able to make connections to things they’ve learned prior and outside applications?

- Through utilizing the perceptual variability principle by directly comparing two examples in a class-discussion setting, will students begin to pay more attention to the deeper structure of problems?
  - Will this make solving problems easier?
• Through the use of multiple cognitive practices, in this case a focus on vocabulary, reflection, and numerous representations, will my students increase their conceptual understanding?

Goals

My goals throughout my action research project are first and foremost to modify my teaching in the best interest of my students. In this process, I want my students to learn with me by always seeking to try new things to understand concepts and to apply these strategies in other settings. One of the main reasons I chose cognitive strategies is because they are not just appropriate in the mathematics classroom. One can take a desire to be more attentive to vocabulary when studying biological, literary or musical terms. Cognitive scientists preach that reflection is a technique that can be applied to becoming more conscious of one’s own learning in any setting, especially teaching (Tovani, 2004). The idea of directly comparing two examples can be used to study irony in an English class, for instance, by comparing the irony in *Oedipus Rex* and *Romeo and Juliet* (Willingham, 2009). My final goal is to seek answers to my research questions and investigate the conceptual understanding changes that appear in my students.

**Literature Review**

There have been thousands of studies that focus on why and how students learn mathematics best. By focusing on the teachers, students, materials, curriculum, setting, and numerous other factors, these studies have regularly focused directly on changing what students can do in the physical sense; they can solve quadratic expressions, use manipulatives to demonstrate completing the square, or simplify algebraic expressions. However, less of the time do we focus on refining the ways students think, in a general sense. My action research project
foci on improving the ways students think, through a concentration on vocabulary, reflection, and comparison, and then to monitor academic changes I see in my classroom.

**Vocabulary**

Vocabulary has long received enormous attention in the school setting. However, the majority of these studies have been conducted at the elementary level, that is, kindergarten through fifth grade (Bay-Williams & Livers, 2009; Blessman & Myszczak, 2001; Rubenstein, 2007; Sullivan, 1988). The reason for this is because there is substantial evidence that the gap that develops, generally between 2nd and 4th grade, between high-achieving and low-achieving students can be accredited to the difference in background knowledge held between the two groups. Many students who come in well off have substantial background knowledge and background knowledge allows people to chunk information, so they can store it easier. For example, if I were to ask you to memorize the letters NB_CF_BL_NC_A_ASO_SN_N_A in ten seconds you would most likely struggle. However, it becomes easier when you see NBC_FBI_NCAA_SOS_NASA. This is because of your background knowledge and background knowledge directly affects one’s ability to conceptualize abstractly (Willingham, 2009). The question then becomes how does a teacher increase students’ background knowledge and one of the easiest ways to do this is to increase their vocabulary (Kovarik, 2010).

Thompson and Rubenstein (2000) state that students must understand mathematics vocabulary if they are expected to master content and be able to apply it to future situations. Also, a common problem among students transitioning to problem solving is putting word problems into mathematical statements and Marzano (2004) found that teaching academic vocabulary could raise standardized test scores by as much as 33%. Recent research by Gifford and Gore (2008) also showed that struggling math students who received increased vocabulary
help saw standardized test scores increase by as much as 93%. Schoenberger and Liming (2001) conducted a vocabulary study on 6th grade students that found an increase in students’ abilities to correctly define and apply terms in abstract problems, identify key components of mathematical statements, including the particular cue words in story problems, and recall the mathematical tools required to solve the problem and label their answers. These findings were prevalent in other papers as well (Bradley, 2003; Miller, 2007; Schwarz, 1999; Sowder, 1979). In searching through these studies however, the biggest thing missing was a focus on vocabulary at the university level.

A study conducted by Sowder (1979) was one of very few to investigate vocabulary at the college level. However, the actual focus of the study was on improving methods for teaching problem solving, one of which was devoting time to mathematical vocabulary and symbolic notation. The researcher concludes after conducting his study that an area requiring further research includes studies examining the role of verbal and symbolic language in the decoding phase of problem solving. By putting a greater focus on vocabulary and its role in pointing out the consequences or implications of key terms in a problem statement, I look to contribute to the lack of knowledge in this area.

**Reflection**

The concept of reflection in terms of cognitive practices is not a new technique and when one searches through the vast collection of studies they almost all pertain to a teacher reflecting upon his or her pedagogy. Although there are some, very few turn that concept around and apply it to their teaching by having the students reflect. However, the ones that do have seen some encouraging results, as we will see.
There are three types of metacognitive knowledge that a teacher can ask a student to reflect upon: 1) strategic knowledge, that is strategies developed by students when learning or asking questions; 2) knowledge about cognitive tasks, that is recognizing different tasks require different skills; and 3) self-knowledge, that is knowledge of one’s own strengths, weaknesses and self-motivation. By making students aware of these differing types of metacognitive knowledge, they will be more likely to apply them in the classroom (Pintrich, 2002).

The reason that a teacher should care about reflection and encouraging students to practice the different types of metacognitive knowledge is that there is ample evidence that students who regularly practice reflection are more capable of solving routine problems, extending the concept and constructing their own knowledge (Narode, 1985; Odafe, 2010; Ozsoy, 2011; Wheatley, 1992; Young, 2011). One of the earliest proponents of reflection in the classroom, John Dewey, stated that through reflection we are “learning more about ourselves and the world in which we live,” (Dewey, 1910, pp. 152). In a study by Ozsoy (2011) that looked at the relationship between metacognition and mathematic achievement, it was found that by having students reflect upon their learning and justify problem solving strategies there was a significant and positive relationship ($r = .648$, $p = .01$) between the two variables of interest.

Despite the advantages of utilizing reflective practices in the classroom, few teachers make reflection an integral aspect and some studies have explained why that may be the case. One of the reasons is that teachers feel constrained to simply get students to a point where they have enough understanding to be reflective, especially when the material is overwhelming. Although it is not enough for students to just complete tasks, as they must also be able to reflect upon the process, this can sometimes prove difficult (Wheatley, 1992). Gustafson and Bennett (1999) discovered that encouraging military cadets to reflect by means of written responses in
diaries was challenging and most did not produce results indicating any deep reflection. Unfortunately, these findings are consistent with other studies that state it is demanding to promote individual reflection (Stamper, 1996). However, where these studies fell short was in failing to have any structure in the reflection process, and this is where I look to make changes. Through a guided-discovery approach, the teacher should ask the questions, verbally or in writing, which can promote worthwhile reflection. What the prior studies found was that it’s difficult for novices to reflect on their own, mostly due to the fact that the students are just learning the material and are not at the point cognitively to know what they should, or could, be looking for in the lesson.

**Multiple Embodiment Principle / Perceptual Variability Principle**

Mathematics education specialist, Zoltan Dienes, coined the term perceptual variability principle, also called the multiple embodiment principle. Dienes was a proponent of constructivism, a view that learning mathematics is a deeply personal affair conducted through individual experiences. He saw the beauty in mathematics and believed children should desire to study the subject for its intrinsic value instead of utilitarian reasons that included passing a class or getting a high-paying job. His specialty was learning theory, and his theory was based off of four principles, where the final three are actually a part of the first. The major and first principle is called the dynamic principle and states learning mathematics must occur in three stages: 1) play stage, 2) structured learning, and 3) formal math lesson. He was most famous for the play stage, where he would use songs, dances, and manipulatives to introduce students to advanced math concepts. Only when students had experienced how the math played out in the real world, through one of the physical senses, could they begin to symbolize it in a formal lesson. The second and third principles are the mathematical variability principle and the constructivity
principle. The mathematical variability principle encourages teachers to show students different representations of the same concept where you alter irrelevant factors. For instance, when teaching about interior angles of a polygon, the teacher should change the size and direction of the shape while noting the important factor, the sum of the interior angles, is the same. The constructivity principle simply states that children must be allowed to develop an intuitive manner for the mathematics. The final principle is the multiple embodiment or perceptual variability principle (Post, 1988; Sriraman & Lesh, 2007).

The multiple embodiment principle states, “conceptual learning is maximized when children are exposed to a concept through a variety of physical contexts or embodiments,” (Post, 1988, pp.9). The basic idea is that the more opportunities a student has to experience a concept in different settings the more likely he will be to accredit the commonalities to the process, even though the embodiments still differ (Bart, 1970). Cognitive scientists call these terms the surface and deep structure of the problem. As an example, consider a pre-calculus student who is asked to solve a word problem involving minimizing the amount of fencing materials to enclose a fixed area. Next, the student is asked to find the maximum volume of a box if he only has a fixed amount of materials. Although one of these problems is about minimizing and the other about maximizing, that is they differ in surface structure, the student should soon realize the deep structure is the process of using the vertex of a function to find either its maximum or minimum, depending on the setting. Ultimately, it is the realization of the similarities that these problems share that is the heart of the lesson and what my study looks to utilize.

Although Dienes’ principles have been around for a number of decades there have not been many studies directly investigating the usefulness of just the perceptual variability principle. Cognitive scientist Daniel Willingham confirms this when he states, “Another strategy
that might help (although it has not been tested extensively) is to ask students to compare
different examples,” (2009, pp. 102). One study by Sisakht, Larki, and Bakhshalizadeh (2010)
found that fourth grade students who were taught fractions through the use of multiple
embodiments scored significantly higher and have a deeper conceptual understanding of the
concept of fractions. Also, they have less difficulties and misconceptions when learning fractions
and working with fraction problems. Gningue (2006) found similar results when investigating
the use multiple embodiments to solve algebraic expressions, as did Barody (1988) with place-
value. The use of multiple embodiments was also found to be successful in a linear algebra
course (Harel, 1989). However like the other studies, the methods usually involved displaying a
number of examples without inviting the class or groups to discuss and digest how the problems
or contexts were similar and different. This use of whole class discussions is something that is
missing from the literature and I hope to investigate.

Each of these three individual cognitive practices, utilizing vocabulary, reflection, and
multiple embodiments, have ample opportunity to be expanded on and add to the literature of
current studies. However, investigations into how these practices would enhance a classroom
when enacted together are completely unknown and what makes my investigation especially
intriguing.

Methods

Participants

This study was conducted in my Math 121 College Algebra course on The University of
Montana campus. The course focuses on the study of functions and their inverses. The learning
goals for the course include:

- Use factoring to solve, find zeros or x-intercepts of polynomial functions.
- Solve linear, quadratic, exponential and logarithmic equations and use them to solve
applied problems.

- Use function notation; identify domain, range, and intervals of increasing/decreasing/constant values.
- Find zeros, asymptotes, and domain of rational functions.
- Evaluate and sketch graphs of piecewise functions and find their domain and range.
- Use algebra to combine functions and form composite functions, evaluate both combined and composite functions and determine their domains.
- Identify one-to-one functions, find and verify inverse functions, and sketch their graphs.
- Graph linear, polynomial, radical, rational, exponential and logarithmic functions.

There were 24 freshmen, 5 sophomores, 4 juniors, and 2 seniors for a total of 34 students enrolled in the course, however 2 ended up dropping. My students’ major programs included wildlife biology, forestry, pre-pharmacy, health and human performance, medical technology and undeclared. We met for a 50-minute period, three days a week, however, one of those days is cut short by 15 for mandatory quizzes. Each day I had to cover approximately one section of material from the course textbook and as I am presenting new material each day there are no days to reflect directly on a prior lesson. Because of the course’s rampant pace there is little time for worthwhile group work, extended projects, or tangent lessons on outside applications. I was aware of the circumstances and tight schedule I was under when I elected to take on my project and this is why I chose research items that can be somewhat easily fit into the tight schedule at hand.

Before beginning my study, I discussed with my class the purpose, goals, and methods of my research project and what each of our roles were as co-participants. I informed my students that their identities would be kept confidential during and after the research project. If for any reason, one of my students wished for me to not use the information I collected from her/him in my research, she/he was welcome to inform me at any time. I explained to them that my utmost goal was to improve my teaching and the quality of their learning.

Design
Under the description given in Costello (2003), this study used an action research design. As defined in Costello, action research:

“involves gathering and interpreting data to better understand an aspect of teaching and learning and applying the outcomes to improve practice. It is a flexible spiral process which allows action (change, improvement) and research (understanding, knowledge) to be achieved at the same time,” (pp. 3).

Key aspects of action research are that it is education-improvement oriented, participatory, reflective, collaborative, systematic and organized, inquisitive, testable, and a spiral process of planning, acting, and reflecting (Kemmis & McTaggart, 1988). I directly conducted my action research project for approximately four weeks, as I engaged in the cyclical process of planning, acting, and reflecting on what I had experienced and then repeated the process. I was one of six instructors for the six sections of this course and asked one of my associates to observe my teaching and act as my collaborator. As all of us met weekly to discuss how the class had been going, verify what was coming up, see what new ideas or strategies others were trying, and organize common assessments, these meetings acted as an integral aspect of my collaboration process.

Procedure

When focusing on implementing vocabulary into teaching, studies have found a variety of methods that are particularly useful. Bay-Williams and Livers (2009) state that the vocabulary should be clearly defined in writing with appropriate visuals or handouts whenever necessary. If the term is used differently in everyday language, that needs to be addressed. For instance, precise in everyday language is a synonymous with accurate, but mathematically, precise means consistent, so something can be precise without being accurate. Also, the vocabulary should be
reviewed throughout a lesson and its consequences pointed out when utilized in a problem. Rubenstein (2007) found that by utilizing open-ended assessment questions that required justification and presenting the etymology of the word, students were more successful with new vocabulary. Finally, it was also found to be effective when students were encouraged to use new words in their journal entries (Thompson & Rubenstein, 2000).

With the findings of these studies in mind, to introduce a greater focus on vocabulary in my classroom I first clearly defined any new words and distinctly marked vocabulary terms from other notes. After defining the word, I verbally shared with the class the origin of the word using Steven Schwartzman’s *The Words of Mathematics: An Etymological Dictionary of Mathematical Terms Used in English* as a guide. Throughout the lesson, I constantly restated when we were using consequences of a vocabulary term, for instance, functions that grow exponentially have a constant percent rate of change. Whenever possible, I conducted lead-in questions or examples in hopes that the class will develop the definition naturally. In assessment, I asked questions that focused on knowing the implications of vocabulary terms. For instance, “If you know two lines are perpendicular, what else can you say mathematically about those lines?” For vocabulary terms I found particularly confusing or important I constructed 45-60 second video clips that described the term and posted them to our course website. Finally, I urged students to use vocabulary terms in their reflections. To measure the effects of the vocabulary implementation, I looked for common themes in their reflective sheets (to be used in reflection), justifications on homework problems, performance on assessment items and word usage in class discussions.

In adding a focus on reflection to my classroom, on the smaller scale, I asked students to constantly justify statements during class discussions. For instance, one student told me that when I had a negative exponent I could flip the side of the fraction the term was on and make the
exponent positive, but she had no idea why you could do that so we discussed this as a class. In homework problems and assessments, students were asked to interpret what their answers mean in a physical sense. Finally, and most formally, students were asked to keep a reflective notebook or journal. At the end of every day, the last five minutes were set aside so students could reflect. Questions I supplied to guide them included (Appendix A):

1) In your own words, what did you learn today?
2) Did you see any connections to other things you've learned?
3) What things from today are you still confused about?
4) Can you draw a diagram to illustrate something you learned today?
5) Was there anything new that really clicked or seemed to make sense?
6) If you had to summarize today's lesson in one statement, it would be . . .

Students were graded merely on participation for their journal entries. To measure the effects of the reflection aspect of my research project, I again noted things I experienced in my own reflective journal that I kept and wrote in following each class period. As expected for an action research project incorporating qualitative techniques, student journals were collected and coded for common themes that appeared. Finally, a survey (Appendix B) to measure students’ perceptions on the usefulness of the reflections as well as the two other techniques was distributed after our unit was complete.

In deciding how to structure and employ Dienes’ perceptual variability principle in my classroom, looking to the literature was difficult because suggestions were scarce. Most studies were at the elementary level, where manipulatives play a large role, and in a short class-period, learner-centered discovery isn’t realistic. Other studies done at a higher education level focused on displaying as many examples as possible, with the hopes that something would stick. My goal
was to dedicate one example per lesson that directly compared two items on their surface and on their deep structure. When applicable, I presented the two problems and first asked students to state how they were similar and different from their presentation. Then after solving the problems, we engaged in a group discussion where I noted how they were similar and different, but now in their deep structure. In some instances I recorded, in writing or through pictures, what was accomplished and then coded these discussions, much as one does a journal. I also reflected on these discussions in my own notebook and looked for applications when students were working on their homework or taking tests or quizzes.

In determining the effects of these three implementations as a whole, I conducted a pretest (Appendix C), using an assessment from prior years, to gauge what students knew before we started the unit. Then, on their final assessment (Appendix D), I looked for improvement not only on their performance and overall conceptual understanding, but on specific items that addressed each of the three cognitive practices I was researching.

Results

Data

I collected data to verify what themes arose in each of my three cognitive practices: multiple embodiment, vocabulary and reflections as well as their overall effect on my students’ conceptual understanding. To measure the improvement in students’ overall conceptual understanding, I conducted a pretest to compare to the regular posttest administered at the end of our unit. The graph in Figure 1 displays the class averages before and after the instruction focusing on multiple cognitive strategies. Also, the scoring breakdown for each posttest question and relevant pretest questions, as the problems vary from semester to semester because of different instructors, can be found in Appendix E.
Figures 2 – 6 below display the improvement in specific assessment questions. Figure 2 pertains to a question about the properties of the logarithmic graph, which was a vocabulary focus in my study. Figure 3 deals with function transformations, which dealt with the multiple embodiment aspect of my teaching as we compared multiple ways to look at transformations: function notation, tables, graphs, and then compared it to the knowledge we held from the vertex form for quadratics. Figure 4 concerns a proportionality question that was a vocabulary emphasis. Figures 5 and 6 contain items about polynomials, which was taught with multiple embodiment strategies, similar in nature to what was discussed for function transformations.
Although I label this a multiple embodiment item because we connected the ideas of linear factors with zeros of the graph, one could most certainly argue it is a vocabulary focus.
To measure the effects that students perceived to occur due to reflections, vocabulary, and multiple embodiment strategies, I conducted a post-unit survey. The results of this survey are displayed in Figures 7-10 below and Appendix F.

**Question 1: How helpful were the reflections in your learning?**

![Question 1 Responses](image)

**Figure 7**

**Question 2: What aspects of the reflections did you find particularly helpful (check all that apply)?** *If there were others not specifically mentioned, fill them in for Other.*
Question 3: If you did not find the reflections to be helpful, why not? What would you have changed?

For responses, see Appendix F.

Question 4: Did you find focusing on the definitions of terms (example: logarithm) to be helpful for solving problems?
Figure 9

Question 5: Did you find comparing similarities and differences between current and prior topics (example: function transformations and vertex form from quadratics) to be useful for your learning?
The results shared thus far are the quantitative measures of my study. The qualitative aspect of the research included coding from my own personal observations and the students’ reflective worksheets/journals (Appendix G and Appendix H).

**Analysis**

Before tying a variety data results together to find themes and similar results, the first item shared in the data section is the overall class improvement on the formal assessment after the cognitive-focused instruction. On the pre-test, the class average was a 21.8%, which was so low because students had not seen a majority of the material before and wrote, “I don’t know” on questions they had no knowledge on how to start. After the 3-4 weeks of instruction that was framed in a cognitive-based structure, the class average increased up to 80.6%, clearly a significant increase. Simply from this it is clear that teaching with multiple cognitive practices tends to increase the most basic of conceptual understanding, that being performance on exam assessments.

1. **Student Reflection**

As I worked through my action research it became evident immediately that reflection was going to play the largest role in my study. It was the most tangible of the three for the students as
vocabulary was only a focus for the 30-60 seconds it took to introduce an item and share its mathematical or etymological properties and the use of the multiple-embodiment feature seemed to the students to be just another natural tool for instruction, which it should. Also, reflection included the other two within it, while vocabulary and multiple embodiment were standalones. By this I mean that it was the student reflections that were my main tool for understanding the connections they were making from the vocabulary and from my multiple embodiment practices.

In the final survey, students said with a 6.58 average on a 1-9 point scale, the reflections were beneficial to their learning. I then went a step further and asked specifically what aspects of the reflection they found useful and obtained the following results.

Collect Thoughts/Let Things Sink In (63.64% Agreed)

Students rated this as a tie for the second biggest reason as to why they believed reflections were useful. Student comments included:

- “Mainly the ‘why’ of each step to solving a problem made it stick in my mind.”
- “It's all coming together for the most part.”
- “I thought they were very helpful, it made me actually stop and really think and get the ideas of that lesson down and understand it more. I really took advantage of it to help me.”

In my own personal reflections I also noted that while students weren’t making some of the multiple embodiment connections verbally in class, either because they were shy or hadn’t acquired the knowledge to yet, they were doing so in their writing. Perhaps letting the lesson simmer allowed this to be possible.
Draw Items/Diagrams To Summarize Lesson (36.36% Agreed)

This was the lowest rated item on usefulness of reflections, but in reading the reflections was one of the most frequently responded to questions. In going through the reflections, drawing diagrams of transformations, graphs of exponentials and logarithms, and polynomials occurred frequently. I noted in coding the reflections that students drew pictures showing \(10^x\) and \(\log(x)\) were inverses reflected across \(y = x\), six varieties of diagrams showing the affect of transformations, four pictures of even and odd functions, two sketches summarizing the general graph of power functions, and countless ones tying in all types of properties of polynomials. One student commented, “I liked how the reflections supported all styles of learning. Ex: visual learners could draw a diagram/table to make connections while others could summarize in words.”

Make Connections Between Current and Prior Topics (63.64% Agreed)

Using the reflections to make connections between what was taught that day and what had been discussed previously was clearly an attempt to tie together reflection and the multiple embodiment tool and students rated this the second best aspect of reflections. Many of the items I could discuss here I will do so in the analysis of the multiple embodiment principle, but there were plenty of examples of students tying new with old, for example vertex form of a quadratic and transformations, generalizing growth rates and doubling time to any type of growth, and properties of quadratics with properties of polynomials (e.g., zeros, bouncing, factors, etc.).

Student comments included:

- “The making connections between topics was the most helpful part of these because it helped put what we were learning into a wider context of the course.”
• “Graphs of functions and transformations is basically what we’ve been doing since the start of the class.”

• “This [function transformations] is similar to vertex form that we knew earlier in the year.”

**Summarize Lesson in Your Own Words (50% Agreed)**

On the reflection sheets I asked students if they could summarize the day’s lesson in their own words what would they say and although it wasn’t used each time, some students responded to it. Student comments included:

• “When you shift a graph you shift the domain and range.”

• “Power functions are related to proportionality.”

• “Polynomials helped us solve a maximum volume problem.”

**Think About Where You Are Still Confused (81.82% Agreed)**

The most overwhelming response to the usefulness of reflections focused on students’ ability to realize where they were still struggling. I noticed that when I began implementing the reflections that I received more emails regarding questions than I had prior. Student comments included:

• “What I liked most about the reflections was that they drew my attention to ideas and concepts I did not understand yet.”

• “It worked great to pull out the topics that I didn't fully understand!”

I’ve previously stated that I thought the student reflections were the most useful part of my study and the main reason is that it allowed me to directly peer into my students’ minds, see the connections they were making and most importantly catch their misunderstandings. Most of the time this was through responses to the question on the reflection sheet that asked what was
still confusing, but even other times I saw students summarizing a concept they though they had learned but they were a little off in their thinking. Because I didn’t have time to directly discuss all of these misconceptions in class, I typed up some of the more important or prevalent ones and my responses to them on my course website. Although I didn’t begin this until the second week of my study, I soon realized how valuable it was for students as some commented in the final survey:

- “I enjoyed that common things found on reflections were brought up during class discussion and walked through.”
- “I like how you collected the more common questions ore statements and answered them in class.”
- “I appreciate that you posted the reflection answers off your site.”

II. Vocabulary

In the follow-up survey conducted after the conclusion of my research study, students responded with an average of 6.63, on a 1 to 9-point scale, when asked how beneficial focusing on vocabulary was for their conceptual understanding (Figure 9). When comparing the two questions on the pre and post-test that concerned vocabulary, students saw class averages improve from 27.8% and 41.5% to 92.7% and 79.4% respectively (Figure 2 and Figure 4).

In my reflections, I specifically noted that focusing on the definition of the logarithmic function allowed students to solve expressions like \( \log(3x) = 2 \) more efficiently and with less rote memorization. By that I mean that when I tutored students from other sections they knew to solve \( \log(3x) = 2 \) they could use the exponential function as the inverse, but a majority didn’t know why, whereas my students jumped directly to \( 10^2 = 3x \) based on the definition of the logarithmic function.
Besides better conceptual understanding, I noted in class discussions that some of my higher-achieving students enjoyed hearing why mathematical terms were called what they are in the math community. For example, I shared that mathematicians call functions that have degree 2 quadratic because of the Latin term quadratum, which means square; so just as squares have area $s^2$, functions that are squared are called quadratic. One student shared, “I really like the random facts about algebra, like where the words come from, etc.”

III. Multiple Embodiment Principle

In the follow-up survey conducted after the conclusion of my research study, students responded with an average of 7.04, on a 1 to 9-point scale, when asked how beneficial employing an aspect of the multiple embodiment principle was for their conceptual understanding (Figure 10). When comparing the three questions on the pre and post-test that concerned the multiple embodiment principle, students saw class averages improve from 18.5%, 30.9%, and 7.7% to 85.7%, 83.7%, and 80.3% respectively (Figure 3, Figure 5, and Figure 6).

In my personal reflections, I noted several instances in which the use of the multiple embodiment principle made an impact. These specifically include generalizing growth rate and half-life problems, connecting transformations and vertex form for a quadratic, linking function notation and effects on the table to graphical function transformation, and tying together all the concepts of polynomials, such as zeros/factors and degree/long-run behavior.

The most overwhelming source of my multiple embodiment data came from the student reflections. To see just how abundant these connections were see Appendix H. To get a sense of the student comments that exemplify this fact, consider the following remarks.

- “Finding power functions is exactly the same process as finding exponential functions.”
• “Graphs of functions and transformations is basically what we’ve been doing since the start of the class.”
• “Transformations is what we did to quadratics.”
• “This [vertical/horizontal shifts of transformations] is similar to vertex form that we knew earlier in the year.”
• “Doubling and half-life are set up the same.”
• “It [logarithms and pH] connected to my chemistry class.”
• “log and ln are basically the same, just with different bases.”

One important thing I want to note is that initially I wanted to construct a similarities and differences chart on my whiteboard when directly comparing two examples in the spirit of the multiple embodiment principle, however after the first attempts it was clear this wouldn’t be feasible within the time constraints of the course and I opted to reflect and record the verbal statements students made in my journal.

Conclusions

When considering the initial research questions, the data clearly indicates that instruction incorporating multiple cognitive practices shows a trend of improving students’ conceptual understanding by increasing their ability to problem solve and make connections across multiple topics. I would also conclude specifically that the cognitive practices of vocabulary, reflection and multiple embodiment principle tended to increase background knowledge, connections to outside topics, attentiveness to the deeper mathematical structure, respectively.

Going into the study, I thought that the multiple embodiment aspect of the instruction would have the greatest impact, but the student reflections truly were the most valuable. I believe this was because they allowed all three cognitive practices to be tied together and myself, the
instructor, to understand what my students did and did not understand. I was initially concerned because prior studies (Gustafson and Bennett, 1999; Wheatley, 1992) indicated that college-aged students do not participate well in reflections, but that was not the case for my students and I credit this to the fact that I structured the reflective sheets instead of leaving them as open-response.

**Discussion**

As far as what I will take from this specific study and incorporate into my future classes, I will continue to tie together these concepts in my teaching. I won’t change what I did for the vocabulary or multiple embodiment principle, but the daily student reflections would be far too time-consuming for a teacher to reflect on every weekend. Instead, I like the idea of having some form of assessment each Friday and a reflective aspect at the end of that assessment so a teacher can glance and get a quick feel for the student’s current standing. These are frequently used at the younger-age levels as exit tickets, but are abandoned for university-level introductory classes.

What future studies could undertake to investigate this concept further would be an extended study. Due to the time frame of the course, I was only able to conduct my research over a 3-4 week time period. An extended study could employ the same methodology I did, just over an entire semester course. If this was done and similar results obtained, a truly experimental or quasi-experimental study could be undertaken to begin establishing true causation of these cognitive practices on student conceptual understanding, either individually, as recent studies are sparse, or collaboratively, as any study is nonexistent.

Although as is the case for most action research studies, we can’t be 100 percent certain that the methods used in our unique microcosm are applicable to the entire education community. However, this study points in the direction that taking an interest in shaping what
and how students think can, hopefully unsurprisingly, increase our students’ ability to think mathematically.
References


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Schwarz, J. C. Vocabulary and its effects on mathematics instruction. (Level 1 - Available online, if indexed January 1993 onward). (62408647; ED439017).


Young, A. E. *Explorations of metacognition among academically talented middle and high school mathematics students*. (870284532; ED519181).
Appendix A

Name: ________________________________________  Week 1 Day 1

5.1 – Logarithms and their Properties

1) In your own words, what did you learn today?


2) Did you see any connections to other things you've learned?


3) What things from today are you still confused about?


4) Can you draw a diagram to illustrate something you learned today?


5) Was there anything new that really clicked or seemed to make sense?


6) If you had to summarize today's lesson in one statement, it would be . . .


7) Any other thoughts:
Appendix B

Survey Regarding the Use of Reflections, A Focus on Vocabulary and Comparing Topics

1. How helpful were the reflections in your learning?

   1  2  3  4  5  6  7  8  9
   (not at all) (somewhat) (very)

2. What aspects of the reflections did you find particularly helpful (check all that apply)? *If there were others not specifically mentioned, fill them in for Other.

   ☐ Collect thoughts/let things sink in
   ☐ Make connections between current and prior topics
   ☐ Think about where you are still confused
   ☐ Draw items/diagrams to summarize lesson
   ☐ Summarize lesson in your own words
   ☐ Other:
   ☐ Other:

3. If you did not find the reflections to be helpful, why not? What would you have changed?

4. Did you find focusing on the definitions of terms (example: logarithm) to be helpful for solving problems?

   1  2  3  4  5  6  7  8  9
   (not at all) (somewhat) (very)

5. Did you find comparing similarities and differences between current and prior topics (example: function transformations and vertex form from quadratics) to be useful for your learning?

   1  2  3  4  5  6  7  8  9
   (not at all) (somewhat) (very)

6. Any Other Thoughts Are Appreciated:
Appendix C

Unit 3 Pre-Assessment

Name: __________________________________________

1. (8 pts) Consider the function \( f(x) = \log(x) \).

(a) Sketch a graph of the function \( f \).
   Make sure you show intercept(s) and asymptote.

(b) Domain of \( f \):

Range of \( f \):

(c) Fill the following blanks for \( f \).
   - When \( x \to \infty \) the values \( f(x) \to \) ______
   - When \( x \to \) _____ the values \( f(x) \to -\infty \)

(d) Which (if any) of your answers in (b) and (c) would change
   if \( f(x) = \ln(x) \) instead?

2. (15 pts) The domain of the function \( y = f(x) \) is \([-2, 3]\). Its graph is given below (and
   a table showing some of its values is given in item (c) below).

\[ y = f(x) \]

(a) (3 pts) If \( g(x) = f(x + 2) \) find \( g(1) = \)

(b) (6 pts) Draw the graph of the function \( g(x) = f(x + 2) \) in the grid provided above.

(c) (6 pts) Write a table for the function \( h(x) = -5f(x) \)

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3. (8 pts) Below graphs of $f$ and $g$ are given. Given that the graph of $g$ is a translation of the graph of $f$, write a formula for $g$ (in terms of $f$).

\[ g(x) = \]

4. (8 pts) Simplify the following expression. (Combine like terms in the numerator, keep the denominator in factored form.)

\[
\frac{x + 1}{x(x - 2)^2} + \frac{5}{x^2(x - 2)} =
\]

5. (10 pts) Newton's second law implies that the force an object experiences is **directly proportional** to the acceleration of the object.

   (a) (4 pts) Write a formula for the force, $F$, as a function of the acceleration, $a$.
   (Your formula will contain a constant of proportionality.)

   (b) (6 pts) If it takes a force of 60 Newtons to accelerate a baseball to 400 m/s$^2$, what is the acceleration of the baseball when a force of 45 Newtons is applied to it?
6. (12 pts) If \( f(x) = 3x^3 - 9x^2 - 30x \)
   (a) Give \( f(x) \) as a product of linear factors.
   (b) Find the zeros of \( f(x) \)
   (c) Find \( f(1) \).
   (d) Which power function has the same end-behavior as \( f(x) = 3x^3 - 9x^2 - 30x \)?
   (e) Sketch a graph of \( f(x) \) showing zeros and end-behavior.

7. (12 pts) Find a possible formula for the polynomial function whose graph is given below.

\[
f(x) =
\]

8. (8 pts) Below are four graphs of power functions. From the list of formulas below, choose one possible formula which best fits each particular graph.

\[
\begin{align*}
y &= \frac{1}{10x^3} & y &= \frac{1}{9x^2} & y &= 8x^3 & y &= 5x^6 \\
y &= -\frac{1}{10x^3} & y &= -\frac{1}{9x^2} & y &= -8x^3 & y &= -5x^6
\end{align*}
\]
9. (10 pts) Consider the rational function \( g(x) = \frac{3x + 3}{(x - 2)(x + 3)} \)

(a) Find all vertical asymptotes (if any).

(b) Find all horizontal asymptotes (if any).

(c) Find all \(x\)-intercepts (if any).

(d) Find the \(y\)-intercept (if any).

(e) \( \lim_{x \to -\infty} g(x) = \) _____ and \( \lim_{x \to \infty} g(x) = \) _____

10. (10 pts) Which function dominates as \( x \to +\infty \)?

Please provide a brief explanation, such as “\( f(x) = 2 + 20x \) dominates \( g(x) = 30 + 3x \) because both are linear and the slope of \( f \) is bigger.”

(a) \( y = 2x^3 \) or \( y = 100x^2 \)?

(b) \( y = e^{0.5x} \) or \( y = 1.5^x \)?

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Score /100
Appendix D

M 121– Fall 2013 – Test 3 — Draft 4 Name: ____________________________

Coordinator: R. Souza Friday, Nov. 15

Show your work in the space provided. At most half-credit is given when work is not shown. If you use a graphical technique, sketch the graph on the test paper. This test has 9 problems in 4 pages. Relax and do your best.

1. (12 pts) Use the properties of exponents and logarithms and show work that solves the following equations. Round your answer to 3 decimal places.

   (a) \(5 \log(2x) - 7 = 3\)

   (b) \(25(3^x) = 5(6^x)\)

2. (10 pts) Radioactive carbon-14 decays according to the function \(Q(t) = Q_0 e^{-0.000121t}\) where \(t\) is time in years. \(Q(t)\) is the quantity remaining at time \(t\), and \(Q_0\) is the amount present at time \(t = 0\). Estimate the age of a skull if 23% of the original quantity of carbon-14 remains.

3. (10 pts) Let \(D_1\) and \(D_2\) represent the decibel ratings of sounds of intensity \(I_1\) and \(I_2\), respectively. Then the decibel ratings are related by the formula

   \[D_1 - D_2 = 10 \log(I_1/I_2)\]

   How many times more intense is the sound of a power saw (which measures 110 decibels) than the sound of a snowblower (which measures 105 decibels)?
4. (12 pts) Consider the graph of \( y = f(x) \) given below.

   (a) Is this the graph of \( y = \log x \) or \( y = \ln x \)?

      How do you know?

   (b) Domain of \( f(x) \): __________________

   Range of \( f(x) \): __________________

   (c) As \( x \to \infty \), \( f(x) \to _____ \)

   As \( x \to _____ \), \( f(x) \to -\infty \)

5. (12 pts) Consider the function \( y = f(x) \) with domain \([-3, 4]\) and range \([-1, 2]\) whose graph is given (twice) below.

   (a) Let \( g(x) = 0.5f(x) \). What is the range of this new function \( g \)?

   (b) Let \( h(x) = f(-x) \). What is the domain of this new function \( h \)?

   (c) (6 pts) Now let \( G(x) = f(x - 1) + 3 \).

      Use the formula of \( G \) and the graph of \( f \) to compute

      \( G(3) = _____ \)

      Graph \( G(x) = f(x - 1) + 3 \) on the grid to the right.
6. (a) (2 pts) Is \( f(x) = \sqrt[3]{8x^5} \) a power function? _____ How about \( g(x) = \frac{12}{x^2} \)? _____

(b) (9 pts) If \( h(x) \) is a power function with \( h(1) = 2 \) and \( h(4) = 32 \), find a formula for \( h(x) \).

7. (10 pts) Driving at 72 mph, it takes approximately 1.5 hours to drive from Missoula to Helena.

(a) The time it takes to drive between these two cities is inversely proportional to the average speed of the trip. Write a formula for the time of the trip \( t \) as a function of the average speed \( v \). Hint: Use the information above to find the constant of proportionality \( k \).

(b) To get to Helena in two hours, what would the average speed have to be?

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Score /100
8. (13 pts) Given the polynomial function \( f(x) = -2(x + 3)(3x - 5)(x - 2) \), find the following:
   
   (a) (2 pts) Degree = 
   
   (b) (3 pts) Zeros:
   
   (c) (2 pts) y-intercept:
   
   (d) (2 pts) Leading term \( a_n x^n \) (power function which gives the end behavior):
   
   (e) (4 pts) Describe the long-run behavior of the polynomial.
      As \( x \to \infty \), \( y \to \) and as \( x \to -\infty \), \( y \to \)

9. (10 pts) Find a possible formula for a polynomial function whose graph is given below.
### Appendix E

#### Test 3 Breakdown

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**StDev Improv** 28.24%  **StDev Improv** 27.66%
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| Avg Improv | 76.36% |
| StDev Improv | 26.75% |
Appendix F

Responses to Question 3

I think the reflections were a good idea and if I was still confused on something they would have helped but most of the stuff we learned was review.

I thought they were very helpful, it made me actually stop and really think and get the ideas of that lesson down and understand it more. I really took advantage of it to help me.

Personally, I didn't pay that great of attention to the reflections.

Responses to Question 6

The making connections between topics was the most helpful part of these because it helped put what we were learning into a wider context of the course.

It's all coming together for the most part.

Mainly the "why" of each step to solving a problem made it stick in my mind.

What I liked most about the reflections was that they drew my attention to ideas and concepts I did not understand yet.

It worked great to pull out the topics that I didn't fully understand!

I enjoyed that common things found on reflections were brought up during class discussion and walked through.

I liked how the reflections supported all styles of learning. Ex: visual learners could draw a diagram/table to make connections while others could summarize in words.

I like how you collected the more common questions ore statements and answered them in class. Although for me, I wouldn't have used the reflections as a learning tool.

I thought this course was well taught and notes and everything were well understood. Thank you!

I appreciate that you posted the reflection answers off your site.
Appendix G

Summaries of Personal Reflections

**Vocabulary Item**
**Multiple Embodiment Principle Item**

**Lesson 1 - Logarithms and their properties**

I realized that through choral responses that the class had an easier time solving logarithmic expressions such as \( \log(2x) = 1.8 \) through the use of the definition of the logarithm than through using inverses, that is \( 10^{\log(2x)} = 10^{1.8} \). In talking with students as they worked on problems, they stated it made more sense than the algorithmic \( (10^{\log}) \) method that those who had seen it before where the most familiar with.

The rest of the lesson focused on introducing the basic properties of logs and due to having to go over the prior test, there wasn’t any time for an explicit multiple embodiment focus.

**Lesson 2 - Logarithms and Exponential Models**

Students were doing a great job of recognizing the similarities between half-life and doubling time when solving for them in general. They noted the original differences in the context of the problem and when we had worked through both in general, they were able to note that it always boiled down to \( \frac{\ln (\text{ratio})}{\text{continuous growth rate}} \). When I extended this to the idea of quarter-time or quadruple-time, they (chorally) stated \( \ln(1/4)/r \) and \( \ln(4)/r \).

**Lesson 3 – The Logarithmic Function, Exponential Function and Inverse**

Very little instruction time today (20-22 minutes) as we had questions from homework to start class, 15-minute quiz, and then 5-minute reflections. The lesson was on the graphs of \( \log(x) \), \( \ln(x) \), \( 10^x \) and \( e^x \). I presented the graphs using my calculator and Geogebra. We discussed how logarithmic functions have a vertical asymptote and exponentials have a horizontal asymptote. We then compared tables for these four functions with their graphs, and arrived at the conclusion that there were two pairs of inverses. On the quiz this day, 24 out of 32 students correctly defined the logarithmic function when asked, “What is a logarithmic function, that is if \( y = \log(x) \) what do you know?” A correct response what that \( y \) was the power of 10 such that \( 10^y = x \).

**Lesson 4 – Vertical and Horizontal Shifts**

Today was successful because students easily connected with vertical and horizontal shifts due to their experiences with vertex form. I began the class by going through examples in the past that we did with vertex form to refresh them. I’ve found that students readily remember things like \( f(x+3) + 2 \) shifts \( f(x) \) to the left 3 and up 2, they seem to “like” these easy rote rules. In extending the graphical display to a table, I found that students struggled at first and I think this is because they still don’t have function notation down. I later verified this guess because although the majority of students did well with the table question on the Friday quiz, 0 missed the graphical
questions for transformations. I’m still having troubles getting much actual class participation for the multiple embodiment aspect of the research (done comparing vertex to any function and comparing tables and graphs in this case, which some students stated in reflections helped tie the concept together) but students do seem to be making these connections in their writing, if not verbally.

**Lesson 5 – Reflections and Vertical Scaling**

Students were not troubled with horizontal and vertical reflections. However, I could tell that there were some worried faces after discussing even and odd functions. I think the problem with this is that students still struggle with function notation (again) so they get graphically that an even function is a symmetric across the y-axis but struggle with the formal definition that \( f(x) = f(-x) \). Perhaps something I could have done is write out in English that “\( f(x) = f(-x) \) says, “the output (y-value) at x is equal to the output (y-value) at \(-x\). The only way this can occur is if we have symmetry across the y-axis.” The same can be said for the definition of odd function. My vocabulary this day was focused on even and odd because the vocabulary in this section is limited and the fact that \( y=x^2 \) and \( y=x^3 \) provide good guides is useful. No multiple embodiment today.

**Lesson 6 – Power Functions**

Today was limited because of questions and the quiz. We got about \( \frac{3}{4} \) the way through the lesson on proportionality and power functions. I made sure to note the difference between exponential functions and power functions because they are easily confused (even one of my better students was confused why \( 6^x \) was not a power function at first). The majority of the lesson was fine (vocabulary was focused on knowing the difference between a power and exponential function as well as what it means to be proportional, that is related but not equal) and no multiple embodiment for this lesson because of time (I wanted to connect the graphical displays to what was shown in tables). Students were visibly having a hard time grasping the graphs of power functions with negative exponents.

**Lesson 7 – Introduction to Polynomial Functions**

Today was introducing polynomials. I shared the vocabulary poly=many and nom=name/term for vocabulary and then wrote the definition of a polynomial. We then used an online Desmos applet to look at the long-run behavior of various polynomials and concluded that the leading term is the deciding factor. There was no opportunity for multiple embodiment since the topic was simply being introduced and the vocabulary of leading term, coefficients, degree and other terms was the focus.

**Lesson 8 – Connecting All Ideas from Polynomial Functions**

Today was a big day because we got to tie in a lot of features from the past and I continually hinted that these procedures were the generalization of what we did with quadratics. We were studying the short-term behavior of polynomial functions by investigating zeros. Using a Desmos applet we saw how graphs that look similar in the long-run could look very different
near the origin. The class concluded that what made them different was their zeros. From their experiences with quadratics the class did very well adopting the idea that zeros tell us linear factors of the polynomial and vice versa. I felt like the class was doing a good job of making the connections and I am the most excited about seeing how they do on this topic. One interesting questions I received was why do we need a stretch factor when developing the equation from the picture, why can’t we just use the zeros only? I thought this was a very intriguing question and after discussing it in class, look forward to students’ performances on that test question.

Lesson 9 – Applications of Polynomial Functions

Today was a short lesson on how polynomials can be used to solve optimization problems and how the calculator can be utilized to find maximums, minimums and zeros of a polynomial. We only had time for this one problem and nothing was particularly enlightening or available for vocabulary. I stressed multiple embodiment by discussing what the physical contexts are represented in the graphical features, for instance, the zero of a particular graph in question displayed what maximum area we could remove from the corners of a piece of paper in order to still be able to fold the paper up to make a box.
Appendix H

Coding Responses From Student Reflections

**Vocabulary Item**

**Multiple Embodiment Item**

**Lesson 1 - Logarithms and their properties**

**Basic definition of a logarithm (multiple students mentioned this)**

**Use of logarithms to solve problems**

- “Isolate your variable before applying logarithm properties.”

**Connection between log and ln**

- “I saw the connections between logs and ln. They are not that different from each other once you understand how they work.” (One of my most struggling students)

- “You can use log or ln to solve for x [in the exponent]. Both can be used invariably because they serve similar functions.”

- “log and ln are basically the same, just with different bases.”

**Connection between log and exponential functions**

- Use of logarithms to solve interest rate problems.

- “Log rules seemed to mirror exponent rules, as far as multiplying and dividing.”

- “Since logs and ln are ways of dealing with exponents, there is a lot of crossover with exponential functions.”

**Knowing the etymology of math words**

- “I really like the random facts about algebra, like where the words come from, etc.”

**Lesson 2 - Logarithms and Exponential Models**

**General Connection Between Doubling Time and Half-Life**

- “When you double something, the doubling time is ln(2)/r.”

- “Half-life and doubling time are very similar.”

- “Very similar concepts; easy to connect the 2 and do the appropriate.”

- “Doubling and half life are very similar.”

- “Doubling and half-life are set up the same.”

- “Doubling time and half-life equations are essentially the same, just one is doubled and one is halved.”
• “For half-life [the ratio] is divide by 2, for doubling time multiply by 2.”
• “Half life and doubling time is essentially the opposite.”
• General expression for any growth or decay time.

Connection Between Doubling Time and Other Times
• To find how long it takes to increase by a factor of 8 it’s just 3 doubling times.

Connection between Logs and pH
• “It connected to my chemistry class.”
• “There were items that connected to my chemistry class.”
• “1/2 life equation for chemistry relates to ½ life formula we figured out in class.”
• “Plenty of chemistry from pH and radioactive decay.”

Lesson 3 – The Logarithmic Function, Exponential Function and Inverse

Inverse Functions Switch Inputs and Outputs
• “Inverses flip inputs and outputs.”
• Inverses are reflections across y = x b/c they flip inputs and outputs.”

Graphically, Inverses Reflect Functions across y = x
• “Inverses are reflections across y=x”
• “Inverses are y=x line reflections”
• “Inverse functions are reflections across y = x”
• “I see the connections between the graphs [inverses] of this and the graphing sections we did earlier in the year.”

Connection/Differences Between log and ln graphically
• “How to tell the difference between log and ln by looking at graph.”

Connection Between Definition of Logarithm and Graphical Properties
• “Asymptotes at log(0) because there is no exponent of 10 taken to a power that equals 0.”

Inverses of Log and Exponential Functions
• “logs are inverses of 10^x and ln is an inverse of e^x”
• “logs and 10^x are always linked to each other”

Applied Logarithmic Scales
• “The decibel and Richter scale are basically the same thing.”
• “The decibels for something may be at 10 db and something at 30 db but that means the 20 db is a 10^2 difference.”
• “Log is orders of magnitude.”
*Lots of pictures showing 10^x and log(x) are reflections of one another across y=x

**Lesson 4 – Vertical and Horizontal Shifts**

**Drawing Graphs of Transformations**
- One student drew a picture of a random parabola, called it h(x) and then drew pictures for h(x-4) and h(x) - 4.
- Another student did a similar drawing, but for the absolute value function.
- Student made up a graph and displayed h(x) + 2
- Drew a graph of f(x) and f(x) + 2
- Drew a graph of f(t), then f(t) + 5 and f(t) – 5
- Drew a graph of h(x) = x^2 and then a graph for h(x-1) + 2 = (x-1)^2 + 2

**Connection in Vertical/Horizontal Shifts from Vertex Form of Quadratics**
- “I have noticed a relationship to vertex form shifting.”
- “Horizontal/vertical shifts relate to vertex form.”
- “I see the connection between shifts and other graphing material.”
- “I see the connections from past vertex forms.”
- “From vertex form I know how to move functions.”
- “I saw connections with when we worked w/the vertex.”
- “Vertex form is very similar.”
- “The vertex form is the same and tells you how to shift the graphs.”
- “This is similar to vertex form that we knew earlier in the year.”

**Connection between How Table and Graph Interact in Transformations**
- “It’s easy to note the connection to just add and subtract from the table.”

**Order of Operations From Arithmetic Applies to Graphical Transformations**
- “Breaking down the problem by order of operations really clicked.”

**Real-World Applications for Transformations**
- “The example using temperature in an office helped.”

**Lesson 5 – Reflections and Vertical Scaling**

**Drawing Graphs of Even and Odd Functions**
- Drew a graph of x^3 (odd) and cos(x) (even)
- Drew a picture of f(x) = x^2 and noted “f(x) = f(-x) is even”
- Student drew a picture of f(x) and then graphs for 2f(x+3)+1 and –f(x-1)-2
- Drew a picture of sin(x) (odd) and cos(x) (even)

**Connection Between Table and Graph**
- “For reflections, you either flip the x or y-values”
• “Table, table, table. When first trying to use all forms of transformations, I was attempting to plug x in and solve but the whole use of tables first clarified the idea.”

Order of Operations From Arithmetic Applies to Graphical Transformations
• “Order of operations when graphing is the same as before.”

Transformations Can be Done to Any Function
• “Graphs of functions and transformations is basically what we’ve been doing since the start of the class.”
• “Transformations of functions reflect changes in the parent function.”
• “Transformations is what we did to quadratics.”

Connection Between Transformations and Domains/Ranges
• “When you shift a graph you shift the domain and range”
• “Reflections change domains and ranges.”

Lesson 6 – Power Functions

Drawings
• Drew a picture of x^2 and x^4 for positive even powers and then drew 1/x for negative odd powers.
• Drew 4 graphs mixing even/odd powers with positive/negative (4 students)

The larger the power, the faster a function grows/falls
• “x^4 grows faster than x^2”
• “As the power of the function rises, the graph grows more quickly” (and drew picture)

Connection Between Directly and Inversely Proportional
• “The relationship between directly and inversely proportional is based on as x gets large, y gets large or as x gets small, y gets small.”
• “I see the connection between Newton’s 2nd law and directly proportional.”
• “The verbal saying of directly and inversely proportional relates to math.”
• “The directly proportional example made sense when applied to a science environment.”

Proportionality Corresponds to Graphs of Power Functions
• “Proportions and how they apply to power functions and their graphs” was something a student said he/she learned.
• “Power functions are related to proportionality”

Although I didn’t even mention the word asymptote, a student noted in her reflections that asymptotes are everywhere in power functions, especially for negative powers.

Lesson 7 – Introduction to Polynomial Functions
Drawings
- Student drew a graph of a quintic and labeled it “x^5 + ax^4 + …”
- Student drew graphs of x^4, x^3, and x^5 (4 students)

Solving for Power Functions is same as solving for Exponentials
- “How solving for power functions are the same as solving for exponentials.”
- “Using logs to find answers to power function is the same as we did before.”
- “Much like solving (finding the equation) for exponents”
- “Very similar to exponentials with in have two points”
- “Finding power functions is exactly the same process as finding exponential functions. Polynomials are simply power functions strung together.”

Leading Term Determines General Shape (Long-Run)
- “The first term in polynomials determines it long term behavior”
- “Their general shape determined by their leading power function”
- “The thing that really clicked for me is being able to look at an equation and be able to tell how the graph is going to look and how its going to act on paper.”
- “Polynomial functions are just multiple power functions that in the long run appear most like their leading term.”
- “The degree determines the graph”

Differences Between Even and Odd Powers (positive)
- Even and odd powers are the same on the right-hand side (x-positive) but opposite on the left-hand side (x-negative): “just take the parabola and make it a different graph. It flips the left side down over the x-axis”

Lesson 8 – Connecting All Ideas from Polynomial Functions

Drawings (Could also be coded as Multiple Embodiment due to graph and equation connections)
- Student drew a picture of a quartic polynomials with a single zero and triple zero and labeled both
- Student drew a graph and then labeled it (x-1)(x+3)^2 noting the affect of multiplicities on the graph and equation
- Drew a picture of a 4th degree and labeled the zeros x = +2 and x=+-1
- Student drew and noted that when a graph is flat it has odd multiplicity (usually 3 in our case)
- Drew multiple cases of bouncing and single zeros (all degree 4 though)
- Drew graph of multiple powers and multiplicities

Using Zeros to Find Polynomial Equation
- “I learned how to find the equation for a polynomial using the zeros of an equation”
- “When solving for polynomials you can find your zeros”
- “It makes sense how you can use the graph (zeros) to find the equation.”
Connection to Quadratics
- “I see we are using the same basic principles as quadratics.”
- “Using factored form and finding the zeros to polynomials”
- “Similarities between quadratic equations when considering short-term behavior”
- “By knowing your linear factors you can find your zeros, just like the sections before (quadratics).”

Connection Between Graphs and Equations
- “Something that really clicked was using the graphs to make equations and equations to then make graphs.”
- “If x = k is a zero of f(x) then the linear factor works backwards and forwards.”

Lesson 9 – Applications of Polynomial Functions

Drawings
- Many drew pictures of the box maximization diagram (given a flat sheet, if we cut corners out what size should the corners be to maximize our volume)

Similarities between Polynomials and Exponential Functions
- “There are a lot of similar concepts as there were with exponential functions, like solving for zeros or finding variables by plugging in a point.”

Polynomials Can Be Used to Solve Real-World Inquiries
- “Polynomials helped us solve a maximum volume problem.”
- “Setting up equations that reflect the scenarios and converting them to polynomial functions.”

Calculator allows you to Find Maximums, Minimums, and Zeros
- Student drew a picture of a difficult polynomial y = x(8.5-2x)(11-2x) and stated you could find max, min and zeros using calc.
- “On the calculator, you can find your max and min for a polynomial”